Notes on integrability in gauge theory and string theory

2009 J. Phys. A: Math. Theor. 42 254001

(http://iopscience.iop.org/1751-8121/42/25/254001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 254001 (40pp) doi:10.1088/1751-8113/42/25/254001

Notes on integrability in gauge theory and string theory

N Dorey

DAMTP, Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road Cambridge, CB3 0WA, UK

E-mail: n.dorey@damtp.cam.ac.uk

Received 26 February 2009, in final form 2 March 2009
Published 9 June 2009
Online at stacks.iop.org/JPhysA/42/254001

Abstract

This paper reviews the emergence of integrability in the context of the AdS/CFT correspondence at an introductory level. In particular, we discuss how planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and free string theory on $AdS_5 \times S^5$ can both be related to integrable systems in one spatial dimension. We determine the spectrum of the model in the limit of long operators/strings.

PACS numbers: 11.25.Tq, 11.30.Ly

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The AdS/CFT correspondence [1] is a remarkable equivalence between two seemingly very different types of theories. On one side of the correspondence we have $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, which is a non-Abelian gauge theory living in flat four-dimensional Minkowski space. On the other, we have IIB superstring theory on the ten-dimensional spacetime $AdS_5 \times S^5$ which in particular contains a massless spin-2 particle corresponding to the graviton. How can two theories living in different dimensions with different fields content be exactly equivalent? As we review below, some of the apparent differences are illusory and, in particular, the global symmetry group of the two theories is actually the same. The answer is also related to the fact that both theories have a dimensionless coupling constant which can be varied continuously. In the case of the $\mathcal{N} = 4$ theory, this is just the gauge coupling g^2, while for string theory on $AdS_5 \times S^5$ it is the so-called 'worldsheet coupling', g_σ^2,

$$g_\sigma^{-2} = (\text{radius})^2 \times \text{string tension}.$$

The AdS/CFT correspondence involves the identification,

$$g^2 N \equiv g_\sigma^{-4}.$$

Thus, when gauge theory is weakly coupled the equivalent string theory is strongly coupled[1] and vice versa. In general, we know very little about strongly coupled large-N gauge theory (or strongly coupled string sigma models) and so it is very hard to test the predictions of the correspondence against first-principles calculations.

Fortunately recent years have seen great progress in overcoming this obstacle. This progress is largely based on the emergence of *integrability* on both sides of the correspondence. We will define what we mean by integrability in a more precise way below, but for now it can be thought of as a new, infinite-dimensional symmetry which emerges in certain limits of gauge theory and string theory. In particular it entails the existence of an infinite tower of hidden conserved charges in both theories. In explicit calculations it is often manifested by the appearance of 'special' solvable models/equations. Two examples we will meet below are the Heisenberg spin chain and the sine-Gordon equation.

These developments have already provided a strong indication that the theories on both sides of correspondence are solvable in the planar limit[2]. They have also provided the first examples of exact results allowing smooth interpolation between gauge theory and gravity regimes. Finally, the most promising aspect of these advances is that integrability is also present in QCD itself in certain limits. Indeed this is where it was first discovered [5].

The purpose of these notes is to describe the emergence of integrability on both sides of the AdS/CFT correspondence at a level accessible to early-stage graduate students. In particular we will focus on the simplest non-trivial cases on both sides of the correspondence, restricting our attention to one-loop perturbative gauge theory and semi-classical string theory. In both cases we will see that an integrable system in one compact spatial dimension emerges. We will also focus on a particular limit where this spatial dimension becomes large (in a sense to be defined). In this limit the spectrum consists of a set of asymptotic states which undergo scattering via a unitary S-matrix. As we review, integrability corresponds to a remarkable factorization property of the S-matrix. To make contact between gauge theory and string theory we will focus on the spectrum of asymptotic states and demonstrate a non-trivial agreement between the two theories.

Along the way we will also review the basic aspects of the AdS/CFT correspondence. However, this is not meant to be a comprehensive introduction to AdS/CFT which is a huge subject with several excellent review articles [2, 3]. Instead this paper is meant to provide some of the necessary background to understand the rapid subsequent developments in AdS/CFT integrability. Some of these developments are described in the other review articles in this volume and we will not even attempt a summary here. Similarly, we will only mention papers which are directly relevant to the basic aspects of the subject covered here and no attempt is made to provide a complete list of references.

The rest of these notes are organized as follows. In section 2 we review basic aspects $\mathcal{N} = 4$ SUSY Yang–Mills, including conformal invariance, the large-N expansion and the AdS/CFT correspondence. After defining integrability in section 3, we focus on its emergence in gauge theory (section 4). Section 5 is devoted to the integrability of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

2. $\mathcal{N}$ = 4 supersymmetric Yang–Mills theory

Like QCD, the $\mathcal{N} = 4$ theory is a non-Abelian gauge theory in $D = 3 + 1$. Here we focus on the theory with gauge group $G = SU(N)$. The theory has the largest possible spacetime

[1] Here we mean that the worldsheet dynamics of a single string is strongly coupled. In contrast, for the cases we study, the string coupling, g_s, which controls the interaction between different strings, will be small.

[2] The planar or 't Hooft limit is reviewed below.

symmetry group consistent with a renormalizable QFT in 3 + 1 dimensions. The bosonic part of the global symmetry group is $SO(4,2) \times SO(6)$. The first factor is the conformal group in four dimensions which includes $SO(3,1)$ group of Lorentz transformations as a subgroup. The second factor is the global R-symmetry group $SO(6) \simeq SU(4)$. These bosonic generators are augmented by a total of 32 component supercharges, which together generate the supergroup $PSU(2,2,|4)$.

The matter content of the theory consists of a single vector multiplet of $\mathcal{N} = 4$ SUSY which includes the following fields,

Fields		$SO(6)_R$
A_μ		$\mathbf{1}$
λ^A_α	$\bar{\lambda}^{\bar{A}}_{\dot{\alpha}}$	$\mathbf{4} \oplus \bar{\mathbf{4}}$
Φ^a		$\mathbf{6}$

Our index conventions are as follows. The gauge field carries a Lorentz vector index $\mu = 0, 1, 2, 3$. Left- and right-handed Weyl fermions carry Lorentz spinor indices, $\alpha, \dot{\alpha} = 1, 2$ respectively. The six real scalars carry an R-symmetry vector indices $a = 1, 2, \ldots 6$ while the fermions are R-symmetry spinors with indices $A, \bar{A} = 1, 2, 3, 4$ for the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SO(6) \simeq SU(4)$, respectively.

Supersymmetry requires that all fields in the vector multiplet are in the same representation of the gauge group. Specifically, all fields are in adjoint representation of $SU(N)$. Thus the real scalar fields Φ^a are Hermitian, traceless $N \times N$ matrices which transform under the gauge group as

$$\Phi^a \to U^\dagger \Phi^a U \qquad U \in SU(N).$$

The Lagrangian of the theory is uniquely fixed by $\mathcal{N} = 4$ SUSY. Here we only give the bosonic terms explicitly,

$$\mathcal{L} = \frac{1}{g^2}\,\mathrm{Tr}_N\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu\Phi^a\mathcal{D}^\mu\Phi^a + \sum_{a>b}[\Phi^a, \Phi^b]^2 + \text{fermions}\right].$$

The only free parameters of the theory are[3] N and g^2.

2.1. Conformal symmetry

In addition to Poincare invariance, the $\mathcal{N} = 4$ theory is also invariant under *dilatations* or scale transformations,

$$D : x_\mu \to \lambda x_\mu$$

which act on the classical fields X of the $\mathcal{N} = 4$ theory as,

$$D : X(x_\mu) \to \lambda^{\Delta_0} X(\lambda x_\mu),$$

where $\Delta_0 = [X]$ is the mass dimension of the field X. Specifically, the various fields in the theory have classical dimensions

$$\begin{aligned} \Delta_0 &= +1 \text{ scalars } \Phi^a \\ &= +\tfrac{3}{2} \text{ fermions } \lambda^A_\alpha, \bar{\lambda}^{\bar{A}}_{\dot{\alpha}} \\ &= +2 \text{ field strength } F_{\mu\nu}. \end{aligned}$$

[3] The theory also has a vacuum angle θ which will play no role in the following.

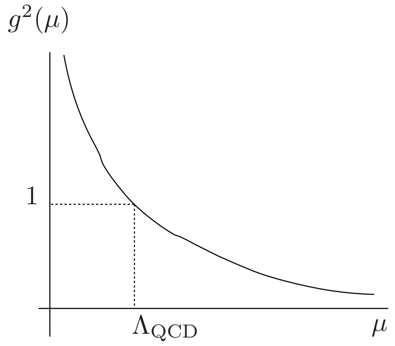


Figure 1. The running coupling.

Each bosonic term in the Lagrangian $\mathcal{L}$ has dimension $\Delta_0 = 4$ and thus the action,

$$S = \int d^4x \, \mathcal{L}$$

is invariant under D.

The spacetime symmetry of the theory includes the following generators,

- The Poincare group is generated by Lorentz boosts $\mathcal{M}_{\mu\nu} = -\mathcal{M}_{\nu\mu}$ and spacetime translations P_μ.
- The $\mathcal{N} = 4$ theory is invariant under the larger *conformal group* $SO(4, 2)$ which also includes the dilatation generator D and the generator K_μ of special conformal transformations

$$K_\mu : x_\mu \to \frac{x_\mu + a_\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}$$

Together these transformations generate $SO(4, 2)$, which is the group of conformal transformations in four dimensions,

$$\left(\begin{array}{cc|c} 0 & D & J^+_\nu \\ -D & 0 & J^-_\nu \\ \hline -J^+_\mu & -J^-_\mu & \mathcal{M}_{\mu\nu} \end{array}\right)$$

where $J^\pm_\mu = (K_\mu \pm P_\mu)/2$. The commutation relations of these generators are given in appendix B.

- The supercharges $Q^A_\alpha, \bar{Q}^{\bar{A}}_{\dot{\alpha}}, S^A_\alpha, \bar{S}^{\bar{A}}_{\dot{\alpha}}$ enlarge the conformal group $SO(4, 2)$ to the $\mathcal{N} = 4$ *superconformal group* denoted $PSU(2, 2|4)$. We will not need any detailed knowledge of the fermionic transformations in the following.

2.2. Conformal field theory

Ordinary Yang–Mills theory and QCD with massless quarks are also conformally invariant at the classical level. But, in these theories, invariance under dilatations D is broken by an anomaly in the quantum theory. The main symptom of this breaking is the non-vanishing of the β function: $\beta(g) = \mu \partial g / \partial \mu < 0$ which leads to a running coupling $g^2(\mu)$ with dependence on the RG scale μ (see figure 1).

In contrast $\mathcal{N} = 4$ SUSY Yang–Mills has

$$\beta(g) \equiv 0.$$

For a perturbative proof see [6]. Arguments for the non-perturbative vanishing of the β-function are given in [7]. As a consequence the dimensionless coupling g^2 does not run and the $SO(4, 2)$ conformal invariance of the classical theory is unbroken.

The natural observables of the theory are *correlation functions of gauge-invariant local operators*. We construct these observables from the following steps,

- Form 'words'

$$\cdots \lambda^A_\alpha \mathcal{D}_\mu \Phi^a \mathcal{D}_\nu \bar{\lambda}^{\bar{A}}_{\dot{\alpha}} \cdots$$

from products (and linear combinations) of all possible 'letters' corresponding to adjoint fields evaluated at the same spacetime point x,

$$\Phi^a(x), \qquad \lambda^A_\alpha(x), \qquad \bar{\lambda}^{\bar{A}}_{\dot{\alpha}}(x), \qquad \mathcal{D}_\mu(x).$$

Note that field strength also arises via the commutator $F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$.

- Make gauge invariants by taking traces. For example, the single trace operator,

$$\hat{O}_1 = \mathrm{Tr}_N\left[\lambda^A_\alpha \mathcal{D}_\mu \Phi^a \mathcal{D}_\nu \bar{\lambda}^{\bar{A}}_{\dot{\alpha}}\right].$$

All such operators have a well-defined classical scaling dimension Δ_0 which is just the sum of the mass dimensions of each 'letter' e.g. $\Delta_0 = 6$ for the operator $\hat{O}_1$ defined above. We can also consider multi-trace operators such as

$$\hat{O}_2 = \mathrm{Tr}_N[\Phi^a \Phi^b]\mathrm{Tr}_N[F_{\mu\nu} F^{\mu\nu}].$$

- Correlation functions of local operators $\hat{O}_i$ at different spacetime points are defined by introducing sources $J_i(x)$ for each operator in the exponent of the path integral in the usual way,

$$Z[\{J_i\}] = \int [\mathrm{d}A][\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\Phi] \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x \mathcal{L} + \sum_{i=1}^{M} J_i(x)\hat{O}_i(x)\right)$$

$$\langle \hat{O}_1(x_1)\hat{O}_2(x_2)\cdots\hat{O}_M(x_M)\rangle = \frac{\delta^M}{\delta J_1(x_1)\delta J_2(x_2)\cdots\delta J_M(x_M)} Z[\{J_i\}].$$

We focus on the simplest case of the two point function of an operator $\hat{O}(x)$ of classical dimension Δ_0. By conformal and translation invariance, the tree-level two-point function is

$$\langle \hat{O}(x)\hat{O}(y)\rangle \sim \frac{1}{(x-y)^{2\Delta_0}}.$$

For example, choose,

$$\hat{O} = \mathrm{Tr}_N[\Phi^n] \rightarrow \Delta_0 = n.$$

The tree-level position space Feynman diagram contributing to the two-point function is shown in figure 3 and consists of n scalar propagators (see figure 2), giving the result

$$\langle \hat{O}_1(x)\hat{O}_2(y)\rangle \sim \left[\frac{1}{(x-y)^2}\right]^n = \frac{1}{(x-y)^{2n}}$$

as expected.

Figure 2. Scalar propagator.

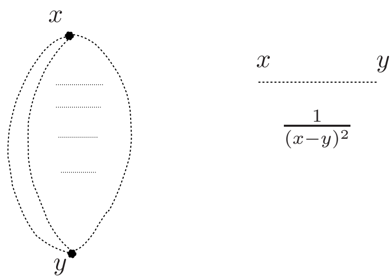


Figure 3. Tree-level contribution to the two-point function.

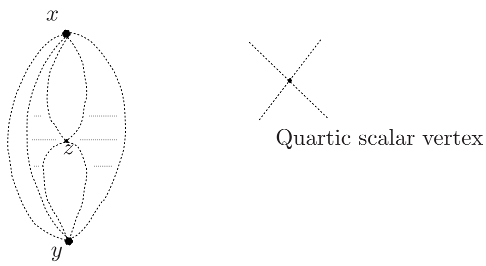


Figure 4. A one-loop contribution to the two-point function.

At the next order perturbation theory we encounter UV divergent contributions to the two-point function such as the one shown in figure 4. This graph includes an integral over the position z of the four-point scalar vertex of the form,

$$\int d^4 z \frac{1}{(x-z)^4} \frac{1}{(y-z)^4}$$

which has a logarithmic divergence when z is near the points x or y.

This reflects a divergence associated with defining the composite operator $\mathrm{Tr}_N[\Phi^n]$ which can be regulated by *point-splitting* e.g. for $n = 2$ defining

$$\hat{O} = \mathrm{Tr}_N[\Phi^2(x)] = \lim_{\varepsilon \to 0} \mathrm{Tr}_N[\Phi(x+\varepsilon)\Phi(x-\varepsilon)].$$

Equivalently we can introduce a UV cutoff $\Lambda \sim 1/\varepsilon$ in momentum space. After subtracting the divergent parts at RG scale μ we define a renormalized operator,

$$\hat{O}_{\mathrm{ren}} = \mathcal{Z} \cdot \hat{O}_{\mathrm{bare}} \qquad \mathcal{Z} = \left(\frac{\mu}{\Lambda}\right)^{\gamma(g^2)},$$

where

$$\gamma(g^2) = \gamma_1 g^2 + \gamma_2 g^4 + \cdots$$

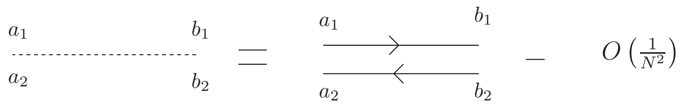


Figure 5. Double-line notation.

is the *anomalous dimension* of the operator $\hat{O}$. Under a dilatation the RG scale changes $\mu \to \lambda\mu$ so that

$$\hat{O}_{\text{ren}} \to \left(\frac{\lambda\mu}{\Lambda}\right)^{\gamma(g^2)} \lambda^{\Delta_0}\hat{O}_{\text{ren}}.$$

Thus the scaling dimension of $\hat{O}_{\text{ren}}$ is,

$$\Delta = \Delta_0 + \gamma(g^2).$$

A conformal field theory is characterized by its spectrum of (renormalized) operators $\{\hat{O}\}$ (as well as the coefficients of the operator product expansion). These operators transform in unitary irreducible representations of the global symmetry group $SO(4,2) \times SO(6)$. States are labelled by Cartan eigenvalues of these representations which are denoted by

$$(\Delta, S_1, S_2, J_1, J_2, J_3).$$

Here $\Delta = \Delta_0 + \gamma(g^2)$ is the scaling dimension, S_1 and S_2 are conformal spins and J_1, J_2, J_3 correspond to three commuting $U(1)R$-symmetries $\subset SU(4)$. Apart from Δ, these charges correspond to generators of compact subgroups of $SO(4,2)\times SO(6)$ and are therefore quantized in integer units.

2.3. The large-N expansion

In QCD, perturbation theory in the gauge coupling $g^2(\mu)$ is only useful in the UV (e.g. deep inelastic scattering). IR physics such as quark confinement and chiral symmetry breaking is non-perturbative in $g^2(\mu)$. In 1979 't Hooft proposed an alternative expansion scheme where the $SU(3)$ gauge group of QCD is replaced by $SU(N)$ and we take the limit,

$$N \to \infty \quad \text{with} \quad \lambda = g^2 N \quad \text{held fixed.}$$

The quantity λ is known as the *'t Hooft coupling*. Corrections to this limit are considered as a power series in $1/N$.

There are several reasons why such an expansion is sensible.

- The leading order $N = \infty$ theory still exhibits confinement and chiral symmetry breaking. This can be established for example by considering the behaviour of the Wison loop area law in lattice gauge theory as a function of N.
- There are many indications in the real world that the large-N approximation might be a good one. These include the occurrence of Regge trajectories in the mesonic spectrum and phenomenological selection rules such as Zweig's rule which are well obeyed in nature.
- The large-N limit leads to important simplifications that *might* allow an analytic solution of the theory.

We will now study the large-N expansion at the level of Feynman diagrams. The expansion works the same in any theory with adjoint fields. We will use the double line notation (see figure 5) which relies on the following decomposition of the adjoint representation of $SU(N)$,

$$\mathbf{adj} \equiv \mathbf{N} \otimes \bar{\mathbf{N}} - \mathbf{1}.$$

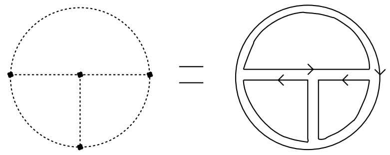


Figure 6. A planar diagram.

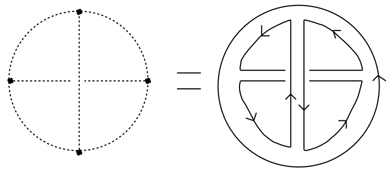


Figure 7. A non-planar diagram.

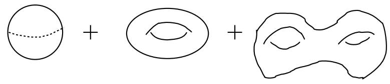


Figure 8. String perturbation theory.

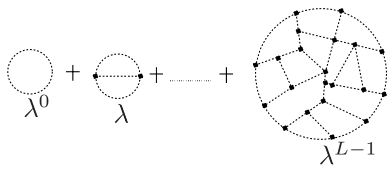


Figure 9. The sum of planar diagrams.

Individual Feynman diagrams can then be thought of as triangulations of an auxiliary 2D surface. Examples are shown in figures 6 and 7. Each edge corresponds to a propagator weighted with $g^2 = \lambda/N$. Each vertex is weighted with $1/g^2 = N/\lambda$. Each face of the triangulation corresponds to a trace over a closed index loop which yields a factor of N. If the total numbers of vertices, edges and faces in a vacuum diagram are $V E$ and F, the diagram scales like

$$N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V}$$

where $\chi = V - E + F$ is the Euler character of the corresponding 2D surface which can also be written as $\chi = 2 - 2g$, where g is the genus.

The $1/N$ expansion is an expansion in the topology of 2D surfaces (see figure 8) which is formally identical to closed string perturbation theory with the identification $g_s = 1/N$. The leading order is given by *planar diagrams* like the one shown in figure 6. The sum over all planar diagrams is still very complicated (see figure 9) and has not yet been evaluated except in various lower dimensional models.

2.4. The AdS/CFT correspondence

The AdS/CFT correspondence is a precise equivalence between two theories,

$$\mathcal{N} = 4 \text{ SUSY Yang–Mills} \equiv \text{IIB} \quad \text{string on} \quad \text{AdS}_5 \times \text{S}^5.$$

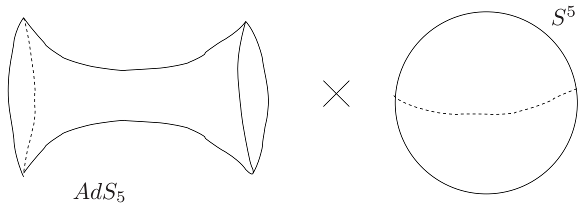


Figure 10. An artists impression of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

2.4.1. Symmetries. To understand the spacetime geometry of the dual string background, see figure 10, note that a 5-sphere S^5 of radius R can be embedded in $\mathbb{R}^6$ as

$$X_1^2 + X_2^2 + \cdots + X_6^2 = R^2.$$

Its isometry group is the rotation group in $\mathbb{R}^6$ which is $SO(6)$. Similarly, five-dimensional anti-de Sitter space of radius R can be embedded in $\mathbb{R}^{4,2}$ as

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + \cdots + Y_4^2 = -R^2.$$

The corresponding isometry group is $SO(4, 2)$. The geometry also admits covariantly constant Killing spinors and is invariant under a 32 component supersymmetry algebra. This is the first point of contact between the theories on both sides of the correspondence. The isometry group $SO(4, 2) \times SO(6)$ of $\mathrm{AdS}_5 \times \mathrm{S}^4$ (+ 32 supercharges) matches global symmetry of $\mathcal{N} = 4$ SUSY Yang–Mills.

2.4.2. Parameters. AdS/CFT also implies the following identifications between the parameters of the two theories:

$$\text{string coupling,} \qquad g_s = \frac{g^2}{4\pi}, \qquad \text{radius,} \qquad \frac{R^2}{\alpha'} = \sqrt{g^2 N} = \sqrt{\lambda}.$$

In the 't Hooft limit, $N \to \infty$ with $\lambda = g^2 N$ fixed, we have $g_s \sim 1/N$. Thus planar gauge theory is mapped to free string theory as expected. Similarly the radius of the geometry in string units is identified according to $R^2/\alpha' = \sqrt{\lambda}$. Thus the geometry is large in string units at $\lambda \gg 1$ when the dual gauge theory is strongly coupled.

2.4.3. Observables. Gauge theory operators of dimension Δ are identified with string theory states of energy Δ. More specifically,

- single trace operators correspond to single string states while multi-trace operators correspond to multi-string states;
- the gauge theory quantum numbers (S_1, S_2) which encode the Lorentz spin of operators correspond to angular momenta of the string on AdS_5. The three commuting R-charges (J_1, J_2, J_3) in the gauge theory correspond to angular momenta of the string on S^5.

Let $\hat{O}_i$, with $i = 1, \ldots, M$, be single-trace operators with scaling dimensions Δ_i. At leading order in the $1/N$ expansion, the dimension for the multi-trace operator, $\hat{O}_1 \hat{O}_2 \cdots \hat{O}_M$ is simply $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_M$, in other words the dimension of a multi-trace operator is simply the sum of the dimensions of its single trace constituents. This corresponds to the fact that the dual string theory becomes free in the planar limit. Energies of multi-string states are just the sum of the energies of the constituent single strings.

The key prediction of the AdS/CFT correspondence is the equality of two spectra:

(A) The spectrum of *dimensions* of gauge-invariant *single-trace* local operators in *planar* $\mathcal{N} = 4$ SUSY Yang–Mills.
(B) The spectrum of *energies* of a *single*, *free* string moving on $AdS_5 \times S^5$.

To check this prediction we need to calculate both spectra and compare. In planar gauge theory, operator dimensions will have an infinite expansion in powers of the 't Hooft coupling λ.

$$\Delta = \Delta_0 + \lambda \Delta_1 + \cdots + \lambda^L \Delta_L + \cdots.$$

The Lth term comes from planar diagrams with L loops. Calculations to fixed order are only reliable for $\lambda \ll 1$. However, in string theory, the spectrum of a free string is determined by quantizing the worldsheet action for string motion on $AdS_5 \times S^5$. Schematically, this has the form of a nonlinear σ-model,

$$S_\sigma = \frac{1}{g_\sigma^2} \int \mathrm{d}^2\sigma \, G_{MN}(X) \partial_+ X^M \partial_- X^N + \text{fermions}.$$

The σ-model coupling constant g_σ^2 is related to the effective string tension which is set by the radius of the geometry in string units (see appendix A),

$$g_\sigma^2 \sim \frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}.$$

Thus the string σ-model is only weakly coupled in the limit of large 't Hooft coupling $\lambda \gg 1$. Thus, as mentioned above, direct gauge theory and string theory calculations are generally only possible in non-overlapping regimes. This makes the predictions of AdS/CFT hard to test for generic observable.

An important exception to this rule is provided by a special set of observables related to *chiral primary operators*. An example of such an operator is

$$\hat{O} = T_{a_1 a_2 \cdots a_M} \, \mathrm{Tr}_N \left[\Phi^{a_1} \Phi^{a_2} \cdots \Phi^{a_M} \right],$$

where $T_{a_1 a_2 \cdots a_M}$ is traceless, symmetric tensor of $SO(6)$. Chiral primary operators saturate a *BPS bound* $\Delta \equiv M$ which implies that their anomalous dimensions vanish for all values of the coupling. The dual states in string theory are the Kaluza–Klein modes of massless SUGRA fields on $AdS_5 \times S^5$. These include the metric, dilaton, axion and the various higher form fields of IIB supergravity

$$g_{\mu\nu}, \qquad \varphi, \qquad C_0, \qquad F_3^{\mathrm{RR}}, \qquad F_3^{\mathrm{NS}}, \qquad F_5^{\mathrm{RR}}.$$

3. What is integrability?

Integrability provides the answer to the question 'When can we hope to solve a dynamical system analytically?'. We start by considering the case of classical mechanics where the notion of integrability can be defined precisely. We consider a dynamical system with M degrees of freedom corresponding to positions, q_i, and conjugate momenta, p_i, for $i = 1, 2, \ldots, M$. We introduce the usual Poisson bracket,

$$\{f, g\} = \sum_{j=1}^{M} \left[\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right]$$

for functions f and g on the phase space. Thus, in particular, we have

$$\{p_i, q_j\} = \delta_{ij}.$$

Time evolution of the dynamical variables is generated by the Hamiltonian $H[p, q] \Rightarrow$ Hamilton's equations,

$$\dot{q}_j = \{H, q_j\} = \frac{\partial H}{\partial p_j}$$

$$\dot{p}_j = \{H, p_j\} = -\frac{\partial H}{\partial q_j}.$$

The basic aim here is to solve Hamilton's equations for arbitrary initial data $p_j(0), q_j(0)$, $j = 1, 2, \ldots, M$. As these are coupled nonlinear differential equations this is usually impossible. Special cases where analytic solutions exist are closely related to the existence of conserved quantities which remain constant along the particle trajectories. For any function $I[p, q]$, $\dot{I} \equiv 0$ if and only if $\{H, I\} = 0$. Thus conserved quantities correspond to functions on phase space which Poisson commute with the Hamiltonian. To illustrate the problem we consider a typical problem of M non-relativistic particles interacting via a pairwise potential $V(X)$,

$$H = \sum_{j=1}^{M} \frac{P_j^2}{2} + \sum_{i>j} V(X_i - X_j). \qquad (1)$$

For a generic choice of interaction potential V, the only conserved quantities are the total energy $E = H$ and the total momentum $P = \sum_{i=1}^{M} P_i$.

A special case arises for dynamical systems with M degrees of freedom which exhibit M independent conserved quantities 'in involution'

$$\Rightarrow \qquad \exists \qquad I_j[p, q] \qquad j = 1, 2, \ldots, M$$

such that,

$$\{H, I_j\} = 0 \qquad \{I_i, I_j\} = 0 \qquad \forall i, j.$$

In this case we have the following theorem.

Theorem. *(Liouville). An integrable system can be solved 'by quadratures'. In other words it can be solved by solving a finite number of ordinary (rather than differential) equations and performing a finite number of integrations.*

In the context of the model of M non-relativistic particles with Hamiltonian (1), integrability arises for certain very special choices for the two-body interaction,

$$V(X) = \frac{1}{X^2}$$

$$= a^2 \frac{1}{\sin^2(aX)}$$

$$= a^2 \frac{1}{\sinh^2(aX)}.$$

The first case defines the Calogero model. The second and third are known as trigonometric and hyperbolic Calogero–Sutherland models respectively.

It is straightforward to generalize the notion of integrability from classical to quantum mechanics. In particular we may quantize an integrable dynamical system by the usual replacements,

$$I_j \to \hat{I}_j, \qquad \{,\} \to \frac{\mathrm{i}}{\hbar}[\,,\,],$$

where $\hat{I}_j$ is a Hermitian operator and $[,]$ denotes a commutator. The resulting system is *quantum integrable* if

$$[\hat{H}, \hat{I}_j] = 0 \qquad [\hat{I}_i, \hat{I}_j] = 0 \qquad \forall i, j.$$

Thus, if the system is quantum integrable, the operators $\hat{I}_j$ for $j = 1, 2, \ldots, M$ can be diagonalized simultaneously with $\hat{H}$. To solve the model we need to find the resulting spectrum of $\hat{H}, \hat{I}_j$. Unfortunately there is no quantum analogue of Liouville's theorem but the spectrum can often be obtained exactly using a set of techniques known as the *Bethe ansatz*.

4. Integrability in gauge theory

In this section we will compute the one-loop anomalous dimensions of an infinite set of single trace operators following Minahan and Zarembo [8].

The matter content of $\mathcal{N} = 4$ SUSY Yang–Mills includes three complex scalar fields,

$$X = \Phi_1 + \mathrm{i}\Phi_2, \qquad Y = \Phi_3 + \mathrm{i}\Phi_4, \qquad Z = \Phi_5 + \mathrm{i}\Phi_6$$

each in the adjoint representation of $SU(N)$. To count insertions of these fields, we define R-charges corresponding to a Cartan subgroup $U(1) \times U(1) \times U(1)$ of $SU(4)_R$,

	J	J_1	J_2
X	+1	0	0
Y	0	+1	0
Z	0	0	+1

Although the results of this section can be generalized to the full operator spectrum, for simplicity we will focus on the $SU(2)$ *sector* of single trace operators made from Z and Y only,

$$\hat{O} \sim \mathrm{Tr}_N \left[Z^{J_1} Y^{J_2} \right]. \qquad (2)$$

To enumerate these operators it is useful to introduce the notation,

$$Z = \uparrow \; Y = \downarrow$$

giving a correspondence between $SU(2)$ sector operators and the configurations of a *spin chain*,

$$\begin{array}{lccccccccccccc} \mathrm{Tr}_N\, [& Z & Z & Y & Y & Z & Z & Z & Z & Y & Z & Y &] \\ & \uparrow & \uparrow & \downarrow & \downarrow & \uparrow & \uparrow & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow\,. & \end{array}$$

The correspondence is as follows,

- An operator of the form (2) corresponds to a configuration of a spin chain of length $L = J_1 + J_2$.
- The classical dimension of the operator is $\Delta_0 = L$.
- The number of flipped spins $M = J_2$.
- We label the sites of the chain with an index $l \in \mathbb{Z}$ with the periodic identification $l \sim L + l$.
- Cyclicity of the trace implies that we must identify spin configurations related by cyclic permutations of the sites.

Following Minahan and Zarembo, our goal is to compute the one-loop anomalous dimensions of all such operators. This is complicated by the fact that operators can *mix* under renormalization. Specifically, the multiplicative renormalization constants required to make correlations finite can have matrix structure[4],

$$\hat{O}^I_{\mathrm{ren}} = \mathcal{Z}_{IJ} \cdot \hat{O}^J_{\mathrm{bare}},$$

[4] NB Global symmetries $\Rightarrow$ $SU(2)$ sector operators do not mix with operators outside this sector.

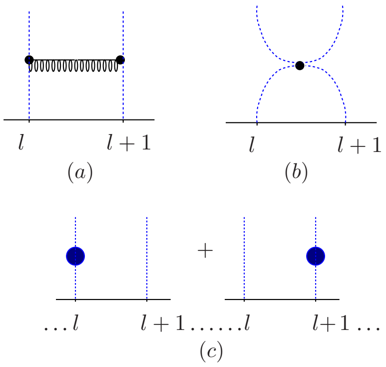


Figure 11. One-loop diagrams contributing to the anomalous dimension. Blob in (*c*) represents self-energy insertion.

where the index I labels all $SU(2)$ sector operators. Equivalently the dilatation generator $D = D_0 + \delta D$, where $D_0 = \mathbb{I}L$ is the tree level piece and δD acts on operators as a non-diagonal matrix with elements,

$$\delta D_{IJ} = \mu \frac{\partial}{\partial \mu} \log \mathcal{Z}_{IJ}.$$

Here we need to find linear combinations of operators with well-defined scaling dimensions. These are the *eigenvectors* of δD_{IJ}. Their anomalous dimensions are the corresponding *eigenvalues*.

We will now sketch the calculation of the one-loop dilatation operator (for more details see [8]),

- The matrix δD_{IJ} can be extracted from the two point function,

$$\left\langle \hat{O}^I_{\text{ren}}(x)\hat{O}^J_{\text{ren}}(y)\right\rangle \sim \frac{1}{(x-y)^{D_{IJ}}}.$$

- The relevant one-loop diagrams contributing to the two point function are shown in figure 11. Note that interactions between non-adjacent scalar propagators are non-planar and are thus suppressed by a factor of $1/N^2$.

The final result of this computation, first performed by Minahan and Zarembo in 2002 is easiest to express as an operator acting on configurations of the spin chain described above,

$$D = \mathbb{I}L + \frac{\lambda}{8\pi^2}\hat{H} + O(\lambda^2),$$

where we express $\hat{H}$ acting on a spin chain as

$$\hat{H} = \sum_{l=1}^{L} (\mathbb{I}_{l,l+1} - \mathbb{P}_{l,l+1}),$$

where $\mathbb{I}$ and $\mathbb{P}$ act on neighbouring spins at sites l and $l+1$ as the identity and permutation operators respectively,

$$\mathbb{I}_{l,l+1}|\cdots \underset{l}{\uparrow} \underset{l+1}{\downarrow} \cdots\rangle = |\cdots \underset{l}{\uparrow} \underset{l+1}{\downarrow} \cdots\rangle$$

$$\mathbb{P}_{l,l+1}|\cdots \underset{l}{\uparrow} \underset{l+1}{\downarrow} \cdots\rangle = |\cdots \underset{l}{\downarrow} \underset{l+1}{\uparrow} \cdots\rangle \, .$$

Here $\mathbb{I}$ is diagonal in $SO(6)$ indices and receives a contribution from each diagram in figure 11. In contrast $\mathbb{P}$ is non-diagonal in $SO(6)$ indices and comes solely from the scalar vertex diagram (figure 11(*b*)).

The problem of finding the one-loop anomalous dimensions then reduces to that of diagonalizing the 'spin-chain Hamiltonian' $\hat{H}$ in other words finding eigenvectors, $\hat{H}|\Psi\rangle = E|\Psi\rangle$. The scaling dimension is then related to the eigenvalue E according to

$$\Delta = L + \frac{\lambda}{8\pi^2}E + O(\lambda^2).$$

Remarkably, the Hamiltonian $\hat{H}$ is the Hamiltonian of the $XXX_{\frac{1}{2}}$ spin chain. This chain was introduced by Heisenberg in 1926 as a simple one-dimensional model of magnetism and first solved by Bethe in 1931. It was understood as an *integrable system* only much later in early 1980s by Faddeev and Takhtajan. To demonstrate integrability it is useful to rewrite the Hamiltonian in terms of $SU(2)$ spin operators $\hat{S}_l^a, a = 1, 2, 3\, l = 1, 2, \ldots, L$, obeying the usual $SU(2)$ commutation relations at each site,

$$\left[\hat{S}_l^a, \hat{S}_{l'}^b\right] = 2\,\mathrm{i}\delta_{ll'}\epsilon^{abc}\hat{S}_l^c. \tag{3}$$

In terms of these operators, the Hamiltonian reads

$$\hat{H} = \frac{1}{2}\sum_{l=1}^{L}\left(1 - \hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_{l+1}\right)$$

with $\hat{\mathbf{S}}_l = \left(\hat{S}_l^1, \hat{S}_l^2, \hat{S}_l^3\right)$.

The algebraic Bethe ansatz due to Faddeev and Takhtajan starts from the existence of a Lax matrix of operators defined at each site,

$$\mathbb{L}_l(u) = \begin{pmatrix} u + \mathrm{i}\hat{S}_l^3 & \mathrm{i}\hat{S}_l^+ \\ \mathrm{i}\hat{S}_l^- & u - \mathrm{i}\hat{S}_l^3 \end{pmatrix}$$

for $l = 1, 2, \ldots, L$ where $u \in \mathbb{C}$ is known as the spectral parameter. A tower of conserved quantities are obtained by constructing the monodromy operator,

$$\begin{aligned} \hat{T}(u) &= \mathrm{tr}_2\left[\mathbb{L}_1(u)\mathbb{L}_2(u)\cdots\mathbb{L}_L(u)\right] \\ &= 2u^L + \hat{q}_2u^{L-2} + \cdots + \hat{q}_{L-1}u + \hat{q}_L. \end{aligned} \tag{4}$$

One may check starting from the commutators (3) that

$$[\hat{T}(u), \hat{T}(v)] = 0 \qquad \forall u, v \in \mathbb{C}$$

$$\hat{H} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}u}\log\hat{T}(u)\bigg|_{u=\frac{\mathrm{i}}{2}} - L\mathbb{I}$$

the operators, $\hat{q}_j, j = 2, 3, \ldots, L$ are conserved, $[\hat{H}, \hat{q}_j] = 0$ and mutually commuting: $[\hat{q}_j, \hat{q}_k] = 0 \forall j, k$. Thus the Heisenberg spin chain is integrable. For more details see [10].

4.1. The Bethe ansatz

In this section we will study Bethe's solution of the Heisenberg spin chain using elementary methods. A useful reference for the material in this section is [9]. It is convenient to organize the spectrum in terms of the number of flipped spins M. For each value of M, we will first consider the case of an infinite chain $L \to \infty$ and then return to the case of finite L. The simplest state corresponds to

M = 0 The ferromagnetic vacuum,

$$|0\rangle = \cdots \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \cdots$$

has eigenvalue $E = 0$. This is consistent with the fact that the corresponding gauge theory operator,

$$\hat{O} = \mathrm{Tr}_N[Z^{J_1}],$$

is a *chiral primary operator* with protected dimension $\Delta \equiv J_1$ (BPS formula: see section 3.5 of [3]). Thus its anomalous dimension must vanish at each loop order.

M = 1 Consider a state with one flipped spin at the lth site,

$$|l\rangle = \cdots \uparrow \quad \uparrow \quad \uparrow \quad \underset{l}{\downarrow} \quad \uparrow \quad \uparrow \quad \uparrow \cdots .$$

The position eigenstate $|l\rangle$ is not an eigenstate of $\hat{H}$. Instead we will try the corresponding momentum eigenstate,

$$|p\rangle = \sum_{l\in\mathbb{Z}} \Psi_p(l)|l\rangle \qquad \Psi_p(l) = \exp(\,\mathrm{i}pl).$$

First one can check that

$$(\mathbb{I}_{l,l+1} - \mathbb{P}_{l,l+1})|l'\rangle = \delta_{l,l'}(|l'\rangle - |l'+1\rangle) + \delta_{l,l'-1}(|l'\rangle - |l'-1\rangle).$$

Thus, acting with the Hamiltonian, we find

$$\hat{H}|l'\rangle = \sum_{l\in\mathbb{Z}} (\mathbb{I}_{l,l+1} - \mathbb{P}_{l,l+1})|l'\rangle = 2|l'\rangle - |l'+1\rangle - |l'-1\rangle$$

and so,

$$\begin{aligned}
\hat{H}|p\rangle &= \sum_{l'\in\mathbb{Z}} \exp(\,\mathrm{i}pl')\hat{H}|l'\rangle \\
&= \sum_{l'\in\mathbb{Z}} \exp(\,\mathrm{i}pl')[2|l'\rangle - |l'+1\rangle| - |l'-1\rangle] \\
&= \sum_{l'\in\mathbb{Z}} \exp(\,\mathrm{i}pl')[2 - \exp(\,\mathrm{i}p) - \exp(-\,\mathrm{i}p)]|l'\rangle \\
&= 4\sin^2\left(\frac{p}{2}\right)|p\rangle
\end{aligned}$$

confirming that $|p\rangle$ is an eigenstate.

It is useful to think of the flipped spin as a particle or *magnon* propagating along the chain with conserved momentum p and *dispersion relation*,

$$E(p) = 4\sin^2\left(\frac{p}{2}\right).$$

The resulting periodicity of momentum space is typical of lattice systems and corresponds to the existence of a Brillouin zone. To return to the case of periodic chain we impose the periodic boundary condition,

$$\Psi_p(l+L) = \Psi_p(l) \rightarrow \exp(\,\mathrm{i}pL) = 1.$$

Thus the allowed values of momentum are quantized,

$$p = \frac{2\pi}{L}n \qquad n \in \mathbb{Z} \tag{5}$$

integer n known as *mode number*.

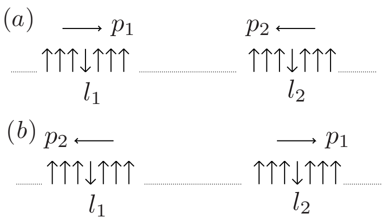


Figure 12. (*a*) Incident magnons and (*b*) transmitted magnons.

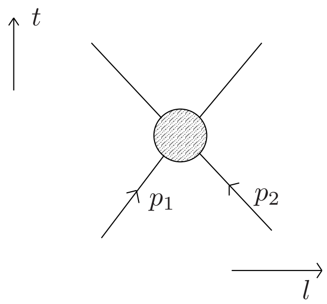


Figure 13. Two magnon scattering.

There is an important subtlety here: we still have to impose invariance under cyclic permutations which implies the identification $|l\rangle \equiv |l'\rangle \forall l, l' \in \mathbb{Z}$. Choosing the momentum p according to (5) we find

$$|p\rangle = \sum_{l=1}^{L} \exp\left(\frac{2\pi n}{L}\right)|l\rangle = \left[\sum_{l=1}^{L} \exp\left(\frac{2\pi n}{L}\right)\right]|l=1\rangle.$$

The sum in brackets vanishes unless $n = 0$. Thus the magnon state does not correspond to a gauge theory operator unless,

$$p = 0.$$

$\mathbf{M = 2}$ For an infinite chain, we can expand a general state in terms of position eigenstates as

$$|\Psi\rangle = \sum_{l_1 \in \mathbb{Z}} \sum_{l_2 > l_1} \Psi(l_1, l_2)|l_1, l_2\rangle$$

where

$$|l_1, l_2\rangle = \cdots \uparrow \;\; \uparrow \;\; \uparrow \;\; \underset{l_1}{\downarrow} \;\; \uparrow \;\; \uparrow \;\; \uparrow \cdots \quad \cdots \uparrow \;\; \uparrow \;\; \uparrow \;\; \underset{l_2}{\downarrow} \;\; \uparrow \;\; \uparrow \;\; \uparrow \cdots \quad .$$

The natural ansatz for wavefunction is a *scattering state* of two magnons,

$$\Psi_{p_1,p_2}(l_1, l_2) = \exp\left(p_1 l_1 + p_2 l_2\right) + \mathcal{S}(p_1, p_2) \exp\left(p_1 l_2 + p_2 l_1\right). \tag{6}$$

The first term on the rhs of (6) corresponds to a partial wave describing two incident magnons propagating as shown in figure 12(*a*). The second term corresponds to partial wave describing two transmitted[5] magnons propagating as shown in figure 12(*b*). A spacetime picture of the scattering process is shown in figure 13.

[5] For identical particles in one spatial dimension the processes of transmission and reflection are indistinguishable. Thus we do not need to include a separate contribution to account for reflection.

As usual in scattering, the coefficients of the incident and transmitted waves can differ by a phase factor $\mathcal{S}(p_1, p_2)$ known as the *S-matrix*.

Reality of the energy eigenvalues requires,

$$\mathcal{S}(p_1, p_2) = \mathcal{S}^{-1}(p_2, p_1).$$

It is straightforward to show that the wavefunction $\Psi_{p_1,p_2}(l_1, l_2)$ given in (6) is an eigenfunction of $\hat{H}$ with eigenvalue,

$$E(p_1, p_2) = 4\sin^2\left(\frac{p_1}{2}\right) + 4\sin^2\left(\frac{p_2}{2}\right) \tag{7}$$

provided the *S*-matrix is given by

$$\mathcal{S}(p_1, p_2) = \frac{u(p_1) - u(p_2) + \mathrm{i}}{u(p_1) - u(p_2) - \mathrm{i}},$$

where $u(p) = \cot(p/2)/2$ is known as the *magnon rapidity*. The eigenvalue (7) is just the sum of the energies of two magnons of momentum p_1 and p_2, respectively. This reflects the fact that the magnons propagate freely except when they reach neighbouring sites with $l_2 = l_1 + 1$.

In any scattering theory an important possibility is that elementary excitations can form bound states. Each such object is a new asymptotic state of the theory with its own dispersion relation and *S*-matrix. Indeed the complete spectrum of the theory in the $L \to \infty$ limit simply consists of all possible free multiparticle states including arbitrary numbers of each species of bound state. In addition to scattering states described above, we can also have a bound state of two magnons corresponding to the pole in the *S*-matrix when

$$u(p_1) = u(p_2) + \mathrm{i} \to \frac{1}{2}\cot\left(\frac{p_1}{2}\right) - \frac{1}{2}\cot\left(\frac{p_2}{2}\right) = \mathrm{i}, \tag{8}$$

which corresponds to a bound state with $U(1)$ charge $J_2 = Q = 2$ and momentum $p = p_1 + p_2$. The corresponding scattering process is shown in figure 14.

We solve these conditions by setting [9, 10]

$$p_1 = \frac{p}{2} + \mathrm{i}v \qquad p_2 = \frac{p}{2} - \mathrm{i}v$$

in (8) which yields $\cos(p/2) = \exp(v)$. This yields a state with energy,

$$E_2(p) = E(p_1) + E(p_2) = 4\sin^2\left(\frac{p}{4} + \mathrm{i}\frac{v}{2}\right) + 4\sin^2\left(\frac{p}{4} - \mathrm{i}\frac{v}{2}\right) = 2\sin^2\left(\frac{p}{2}\right). \tag{9}$$

Thus the position of the pole uniquely fixes the dispersion relation of the bound state.

Finally, returning to the case of a finite chain of length L we must impose periodic boundary conditions,

$$\Psi_{p_1,p_2}(l_1 + L, l_2) = \Psi_{p_1,p_2}(l_1, l_2 + L) = \Psi_{p_1,p_2}(l_1, l_2). \tag{10}$$

As,

$$\Psi_{p_1,p_2}(l_1, l_2) = \exp(p_1 l_1 + p_2 l_2) + \mathcal{S}(p_1, p_2)\exp(p_1 l_2 + p_2 l_1)$$

$$\Rightarrow \Psi_{p_1,p_2}(l_1 + L, l_2) = \exp(\mathrm{i}p_1 L)\exp(p_1 l_1 + p_2 l_2) + \exp(\mathrm{i}p_2 L)\mathcal{S}(p_1, p_2)\exp(p_1 l_2 + p_2 l_1)$$

and (10) holds provided,

$$\exp(\mathrm{i}p_1 L) = \mathcal{S}(p_1, p_2), \qquad \exp(\mathrm{i}p_2 L) = \mathcal{S}(p_2, p_1). \tag{11}$$

The above equations impose quantization conditions on the two magnon momenta. Note that the quantization condition now involves the two-body *S*-matrix. The resulting transcendental equations are the simplest example of the Bethe ansatz equations we will consider in generality below.

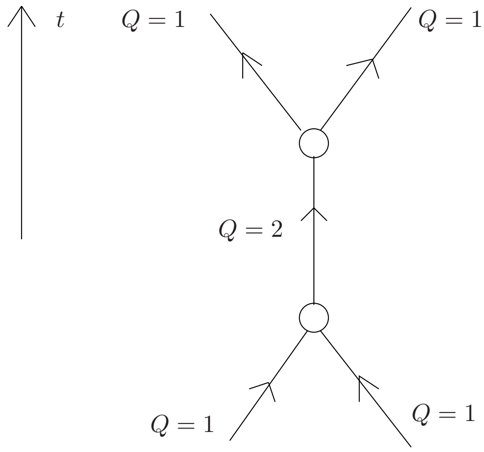


Figure 14. Formation of a bound state in the s-channel.

$M > 2$. In the same way we can expand a general state in terms of position eigenstates for M flipped spins,

$$|\Psi\rangle = \sum_{l_1 \in \mathbb{Z}} \sum_{l_2 > l_1} \cdots \sum_{l_M > l_{M-1}} \Psi(l_1, l_2, \ldots, l_M) |l_1, l_2 \cdots l_M\rangle.$$

A natural generalization of the two-magnon scattering eigenstate is

$$\Psi_{p_1, p_2, \ldots, p_M}(l_1, l_2, \ldots, l_M) = \sum_{\sigma \in S_M} \mathcal{S}^{(M)}_{\sigma}(p_1, \ldots, p_M) \exp(p_1 l_{\sigma(1)} + p_2 l_{\sigma(2)} + \cdots + p_M l_{\sigma(M)}) \tag{12}$$

which corresponds to a scattering state for M magnons with momenta $p_1, p_2, \ldots, p_M$. Here the wavefunction is a sum over all possible partial waves labelled by the permutations σ of the integers $\{1, 2, \ldots, M\}$. Each partial wave is weighted by a corresponding phase $\mathcal{S}^{(M)}_{\sigma}(p_1, \ldots, p_M)$ which can be thought of as an 'M-body S-matrix'.

For a generic system the M-body S-matrix is not related in any simple way to the two-body S-matrix $\mathcal{S}(p_1, p_2)$ given above. However for the Heisenberg spin chain we find a key simplification:

The multi-magnon scattering amplitude factorises exactly as a product of two-magnon scattering amplitudes.

Example. For $M = 3$, considering the scattering of three magnons with momenta $p_1 > p_2 > p_3$ shown in figure 15. This scattering corresponds to the three-body S-matrix $\mathcal{S}^{(3)}_{\sigma}(p_1, p_2, p_3)$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

The statement of exact factorization is

$$\mathcal{S}^{(3)}_{\sigma}(p_1, p_2, p_3) = \mathcal{S}(p_2, p_3)\mathcal{S}(p_1, p_3)\mathcal{S}(p_1, p_2),$$

where

$$\mathcal{S}(p_1, p_2) = \frac{u(p_1) - u(p_2) + \mathrm{i}}{u(p_1) - u(p_2) - \mathrm{i}}$$

with $u(p) = \cot(p/2)/2$ as above. This is illustrated in figure 16 below.

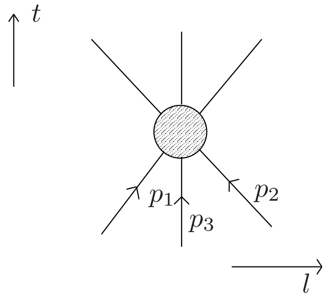


Figure 15. Three magnon scattering.

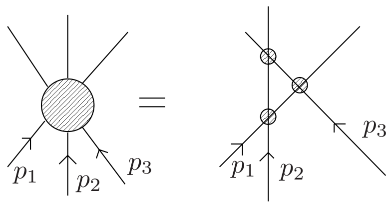


Figure 16. Factorization of three magnon scattering.

A similar factorization property holds for all M. The decomposition corresponds to the fact that any permutation $\sigma \in S_M$ can be factored as a composition of transpositions of adjacent elements. Each transposition corresponds to a factor of the two body S-matrix $\mathcal{S}$. For more details see [9]. In conclusion,

- M-body wavefunction (12) is uniquely determined in terms of the momenta $\{p_1, \ldots, p_M\}$.
- Corresponding energy eigenvalue is just the sum of individual magnon energies,

$$E(p_1, \ldots, p_M) = \sum_{j=1}^{M} 4\sin^2\left(\frac{p_j}{2}\right).$$

We may use this factorization property to find bound states of Q magnons for any Q. These states correspond to poles in the Q magnon S-matrix and therefore to poles in the two-body factors. Using the factorization property, these appear when the momenta of the Q constituent magnons satisfy [10, 11],

$$u(p_j) - u(p_{j+1}) = \mathrm{i} \tag{13}$$

for $j = 1, 2, \ldots, Q-1$. This corresponds to a string of roots in the complex u plane. The condition is easily solved and leads directly to the bound-state dispersion relation:

$$E_Q(p) = \frac{4}{Q}\sin^2\left(\frac{p}{2}\right). \tag{14}$$

Roughly speaking, the Q-magnon bound state corresponds to a state of the spin chain with Q flipped spins where the wavefunction is strongly peaked on configurations where all the flipped spins are nearly adjacent in the chain

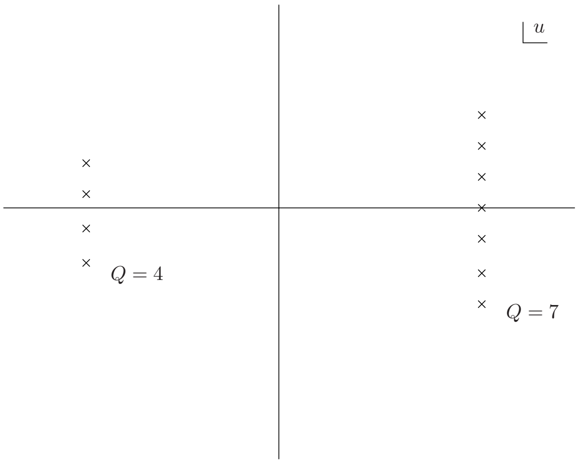


Figure 17. Two Bethe strings corresponding to bound states of charge $Q = 4$ and $Q = 7$, respectively.

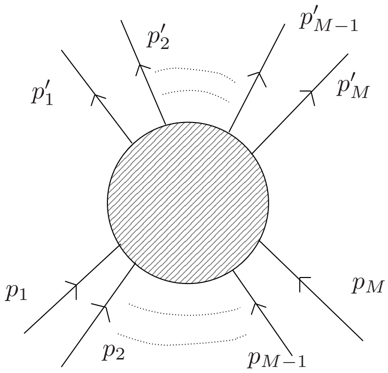


Figure 18. Multi-particle scattering.

4.2. *Integrability and factorized scattering*

Consider M-particle scattering in generic $(1+1)$D theory with dispersion relations $E(p)$ e.g.,

$$\begin{aligned} E(p) &= \frac{p^2}{2m} \text{ non-relativistic} \\ &= \sqrt{m^2 + p^2} \text{ relativistic} \\ &= 4\sin^2\left(\frac{p}{2}\right) \text{ spin-chain.} \end{aligned}$$

Let the incoming momenta be $p_1, p_2, \ldots, p_M$ and the out going momenta be $p'_1, p'_2, \ldots, p'_M$ as shown in figure 18.

Typically we have only two conservation laws: conservation of momentum,

$$\sum_{j=1}^{M} p_j = \sum_{j=1}^{M} p'_j$$

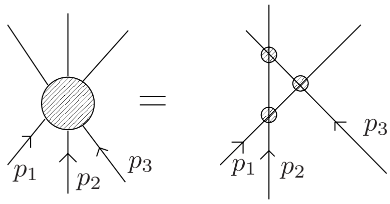


Figure 19. Factorization of multi-particle scattering.

and conservation of energy,

$$\sum_{j=1}^{M} E(p_j) = \sum_{j=1}^{M} E\left(p'_j\right).$$

Thus we typically have two equations for the M unknowns p'_j. For $M > 2$, we expect to find a final-state phase space corresponding to the continuous degeneracy of solutions of these equations. In two dimensions the case of two-body scattering, $M = 2$, is special. Here we have two equations

$$p_1 + p_2 = p'_1 + p'_2,$$
$$E(p_1) + E(p_2) = E(p'_1) + E(p'_2)$$

for two unknowns with two isolated solutions $\{p'_1, p'_2\} = \{p_1, p_2\}$. In other words, the set of outgoing momenta is the same as the set of incoming momenta and therefore the *individual* particle momenta are conserved.

In general this is no longer true for $M > 2$. However if we have *factorized scattering* the situation changes. Suppose the M-body scattering amplitude factorizes into a product of two body amplitudes as shown in figure 19. For reasons described above, individual momenta are conserved in each two-body process. Thus individual momenta are conserved in full M-body scattering,

$$\{p'_1, p'_2, \ldots, p'_M\} = \{p_1, p_2, \ldots, p_M\}.$$

This implies the existence of M conserved quantities. For the spin chain $M \leqslant L$ so the total number of such quantities is L which indicates the integrability of the model. In general, for integrable systems of massive particles on an infinite line,

$$\text{Integrability} \quad \Leftrightarrow \quad \text{factorized scattering}.$$

4.3. The Bethe ansatz equations

To complete the analysis for a finite chain of length L, we must impose periodic boundary conditions on the M-magnon wavefunction (12) under $l_j \to l_j + L$ for $j = 1, 2, \ldots, M$. Like $M = 1, 2$ cases discussed above we get *quantization conditions* for the magnon momenta. In the $M = 2$ case we found (11),

$$\exp(\,\mathrm{i}p_1 L) = \mathcal{S}(p_1, p_2), \qquad \exp(\,\mathrm{i}p_2 L) = \mathcal{S}(p_2, p_1).$$

The M-body generalization is known as the *Bethe ansatz equation* (BAE),

$$\exp(\,\mathrm{i}p_j L) = \prod_{k \neq j}^{M} \mathcal{S}(p_j, p_k). \tag{15}$$

As we have M equations for M unknowns p_j we generically find isolated solutions.

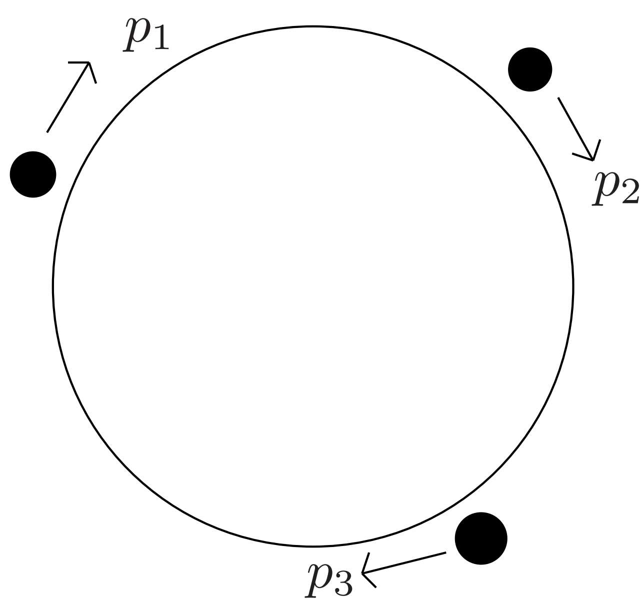


Figure 20. Periodic boundary conditions.

A heuristic derivation of the BAE is as follows. Consider M particles on a circle of circumference L. The $M = 3$ case is illustrated in figure 20. By shifting the position of the jth particle as $l_j \to l_j + L$ we are transporting the particle once around the circle. In the absence of the remaining particles this transport would produce a phase $\exp(\,\mathrm{i}p_jL)$. However, for $M > 1$ the particle must scatter once with each of the other particles as it goes round the circle picking up factors $\mathcal{S}(p_k, p_j)$ for each $k \neq j$. Setting the total resulting phase factor to unity yields the BAE (15).

We are now ready to translate back to the language of gauge theory operators and summarize the full solution for the one-loop dimensions in the $SU(2)$ sector.

- Eigenstates of the one-loop dilatation operator are built from linear combinations of the basis operators,

$$|l_1, l_2, \ldots, l_M\rangle = \mathrm{Tr}_N[\cdots Z\ \ Z\ \ \underset{l_1}{Y}\ \ Z\ \ Z\cdots Z\ \ Z\ \ \underset{l_2}{Y}\ \ Z\ \ Z\cdots Z\ \ Z\ \ \underset{l_M}{Y}\ \ Z\ \ Z\ldots]$$

- The relevant linear combination is the M-magnon state,

$$|p_1, p_2, \ldots, p_M\rangle = \sum_{l_1\in\mathbb{Z}}\sum_{l_2>l_1}\cdots\sum_{l_M>l_{M-1}} \Psi_{p_1,p_2,\ldots,p_M}(l_1, l_2, \ldots, l_M)|l_1, l_2, \ldots, l_M\rangle$$

  with wavefunction $\Psi_{p_1,p_2,\ldots,p_M}$ given in (12).
- The total energy of this state is

$$E = \sum_{j=1}^{M} 4\sin^2\left(\frac{p_j}{2}\right).$$

- The momenta p_j are constrained by the BAE equations (15),

$$\exp(\,\mathrm{i}p_j L) = \prod_{k\neq j}^{M} \mathcal{S}(p_j, p_k). \tag{16}$$

- We must also impose the constraint coming from cyclicity of the trace. This is simply the statement that the total momentum vanishes

$$P = \sum_{j=1}^{M} p_j = 0 \mod 2\pi. \tag{17}$$

- After solving these conditions and evaluating the energy, the one-loop anomalous dimensions of the corresponding operator is $\gamma = \lambda E/8\pi^2$.

4.4. Beyond one loop

So far we have only discussed the spectrum of the theory at one loop and only in the $SU(2)$ sector where operators are formed from the two scalars Y and Z. However, there is by now an abundance of evidence that integrability and the spin chain description persists in the full quantum theory. At least in the limit of a very long spin chain/operator the problem of finding exact anomalous dimensions amounts to finding the exact spectrum of asymptotic states and the two-body S-matrix describing their scattering. These ingredients can then be used to write down asymptotic Bethe ansatz equations (ABAE) which replace those of the one-loop theory described above. Remarkably this problem has been completely solved and the resulting set of ABAE, though conjectural, have passed many non-trivial tests. We will not attempt to review these developments here, but only comment on the form of the higher order corrections to the one-loop results for the $SU(2)$ sector derived above.

It turns out that supersymmetry yields powerful constraints on the magnon dispersion relation and two-body S-matrix [19]. These constraints provide confirmation for an earlier proposal [18] for an exact Bethe ansatz in the $SU(2)$ sector. As before the energy of an M-magnon state is the sum of the energies of individual magnons. However, the exact magnon dispersion relation now reads

$$E(p) = \frac{8\pi^2}{\lambda}\left[\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - 1\right] \tag{18}$$

it is easy to see that this coincides with the one-loop result $E(p) = 4\sin^2(p/2)$ to leading order in the 't Hooft coupling λ. The two-body S-matrix which enters the exact Bethe ansatz equations for the $SU(2)$ sector now takes the form

$$\mathcal{S}(p_k, p_j) = \frac{u(p_k) - u(p_j) + \mathrm{i}}{u(p_k) - u(p_j) - \mathrm{i}} \times \mathcal{S}_D(p_k, p_j), \tag{19}$$

where the rapidity function $u(p)$ is now corrected to

$$u(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}. \tag{20}$$

The quantity $\mathcal{S}_D$ is a 'dressing factor'. Its explicit form has been determined in [20]. The only fact we will use is that the dressing factor does not cancel the S-matrix pole which appears in (19) when, $u(p_k) - u(p_j) = \mathrm{i}$.

An obvious question is what happens to the Q-magnon bound states and their dispersion law (14) described above when we move away from the weak coupling. As the S-matrix pole survives we may use its position to determine the exact dispersion relation. The result is [23]

$$E_Q(p) = \frac{8\pi^2}{\lambda}\left[\sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - Q\right]. \tag{21}$$

This formula clearly reduces to the dispersion relation (14) of the Heisenberg spin chain at the weak coupling. Setting $Q = 1$ we obtain the exact magnon dispersion relation (18). As in the $Q = 1$ case, the bound state dispersion relation (21) can also be regarded as a consequence of supersymmetry. We now verify the formula explicitly in the case $Q = 2$.

For magnon momenta p_1 and p_2 the new pole condition reads[6]

$$\frac{1}{2}\cot\left(\frac{p_1}{2}\right)\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p_1}{2}\right)} - \frac{1}{2}\cot\left(\frac{p_2}{2}\right)\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p_2}{2}\right)} = \mathrm{i} \tag{22}$$

as before we set

$$p_1 = \frac{p}{2} + \mathrm{i}v \qquad p_2 = \frac{p}{2} - \mathrm{i}v$$

and solve for the bound-state momentum $p = p_1 + p_2$ as a function of v. After some computation we obtain a sixth-order polynomial equation, $P_6(t) = 0$, in $t = \cos(p/2)$ with coefficients polynomial in $\exp(v)$ and $a = \lambda/4\pi^2$. The polynomial $P_6(t)$ can be factored exactly into the product of a quadratic $P_2(t)$ and a quartic $P_4(t)$ which are conveniently given as

$$P_2(t) = a(\mathrm{e}^{2v} - 1)^2(1 + \mathrm{e}^{2v} - 2\,\mathrm{e}^{v}t)^2 - 4\mathrm{e}^{2v}(1 + 6\mathrm{e}^{2v} + \mathrm{e}^{4v} - 4\mathrm{e}^{v}t - 4\mathrm{e}^{3v}t)$$

$$P_4(t) = a(1 + \mathrm{e}^{2v} - 2\mathrm{e}^{v}t)^2(t^2 - 1) + 4\mathrm{e}^{v}(t + \mathrm{e}^{2v}t - \mathrm{e}^{v}(1 + t^2)). \tag{23}$$

The physical root is singled out by its weak-coupling behaviour $t = \exp(v)$ needed for agreement with the corresponding formula for the Heisenberg spin chain discussed above. Taking the limit $a \to 0$, one may easily check that the physical root belongs to the quartic equation $P_4(t) = 0$ rather than the quadratic.

The next step is to extract the physical root of the quartic $P_4(t) = 0$, use it to eliminate v in the energy formula,

$$\begin{aligned} E_2(p) &= E(p_1) + E(p_2) \\ &= \frac{8\pi^2}{\lambda}\left[\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} + \mathrm{i}\frac{v}{2}\right)} + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} - \mathrm{i}\frac{v}{2}\right)} - 2\right] \end{aligned}$$

and compare with the predicted dispersion relation (21) for the $Q = 2$ case. A necessary and sufficient condition for agreement with (21) is that the physical root of the quartic should also obey the corresponding energy conservation equation,

$$\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} + \mathrm{i}\frac{v}{2}\right)} + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} - \mathrm{i}\frac{v}{2}\right)} = \sqrt{4 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}. \tag{24}$$

Squaring this equation twice and rewriting it in terms of $t = \cos(p/2)$, $\exp(v)$ and $a = \lambda/4\pi^2$ we obtain the same quartic equation $P_4(t) = 0$, with P_4 as in (23) and we are done. As for the Heisenberg spin chain, the multi-particle S-matrix has a pole corresponding to a Q-magnon bound state for each Q when the condition (13) is satisfied. In principle we could check our proposed dispersion relation (21) for $Q > 2$ by solving this condition, but we will not pursue this here.

[6] The following calculation can be significantly simplified by a change of variables to $x^{\pm} = x(u \pm 2\pi\,\mathrm{i}/\sqrt{\lambda})$ with $x(u) = u + \sqrt{u^2 - 4}$.

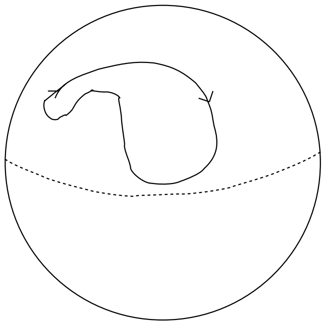


Figure 21. A string moving on $S^2 \subset S^3$.

5. Integrability in string theory

The IIB string on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is described by Metsaev–Tseytlin action [12] which has the schematic form,

$$S_\sigma = \frac{1}{g_\sigma^2} \int \mathrm{d}^2\sigma\, G_{MN}(X)\partial_+ X^M \partial_- X^N + \text{ermions},$$

where G_{MN} is the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric. The coupling constant g_σ^2 is related to the effective string tension which is set by the radius of the geometry in string units,

$$g_\sigma^2 \sim \frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}.$$

The string spectrum can be analysed using semiclassical methods at large 't Hooft coupling, $\lambda = g^2 N \gg 1$. Here we will study the spectrum of classical solutions for a string moving on an $\mathbb{R} \times S^3$ subspace of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

- The $\mathbb{R}$ factor corresponds to the global time of AdS_5.
- The string does not move in the other four spacelike directions of AdS_5 and therefore does not carry the AdS_5 angular momenta S_1, S_2 which correspond to conformal spin in the gauge theory dual.
- The string moves in an S^3 subspace of S^5 and therefore carries only two of the three angular momenta (J, J_1, J_2) which correspond to commuting $U(1)_R$ symmetries on the gauge theory side. We will take this to be the charges J_1 and J_2.

We will see that the corresponding string states can be related to the $SU(2)$ sector operators discussed above. We introduce space and timelike worldsheet coordinates:

$$\sigma \sim \sigma + 2\pi, \tau.$$

The timelike coordinate in spacetime is $X_0(\sigma, \tau) \in \mathbb{R}$ and we immediately fix static gauge by setting $X_0 = \kappa\tau$. The coordinates on the sphere correspond to a unit vector in $\mathbb{R}^4$

$$\mathbf{X}(\sigma, \tau) = (X_1, X_2, X_3, X_4) \qquad \text{with} \qquad |\mathbf{X}|^2 = 1.$$

With the static gauge $X_0 = \kappa\tau$ the string energy is $\Delta = \sqrt{\lambda}\kappa$ and the worldsheet action for a bosonic string is,

$$S_\sigma = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{d}\sigma\, \mathrm{d}\tau [\partial_\alpha \mathbf{X} \cdot \partial^\alpha \mathbf{X} + \Lambda(|\mathbf{X}|^2 - 1)],$$

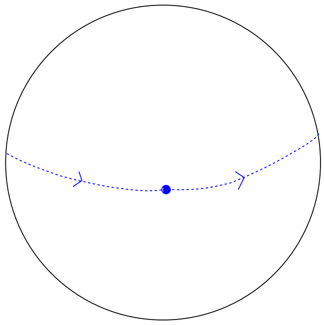


Figure 22. The BMN ground state.

where we introduce the index $\alpha = \sigma, \tau$ for the worldsheet coordinates and $\Lambda(\sigma, \tau)$ is a *Lagrange multiplier* field for the constraint $|\mathbf{X}|^2 = 1$ which defines the S^3 target space. It is also convenient to rescale the worldsheet coordinates as

$$(t, x) = (\kappa\tau, \kappa\sigma)$$

and define the corresponding rescaled light-cone coordinates,

$$x_\pm = \frac{1}{2}(t \pm x), \qquad \partial_\pm = \frac{\partial}{\partial x_\pm}.$$

In static gauge $X_0 = \kappa\tau$ the energy density along the string is constant. In the rescaled coordinate x the energy density is $\sqrt{\lambda}/2\pi$.

Conserved Noether charges for the string motion correspond to angular momenta on the sphere. Here we define a $U(1)_1 \times U(1)_2$ Cartan subgroup of the $SU(2)_L \times SU(2)_R$ isometry group of the target space under which the complex coordinates $Z_1 = X_1 + \mathrm{i}X_2$ and $Z_2 = X_3 + \mathrm{i}X_4$ have charges $(1, 0)$ and $(0, 1)$ respectively. String states carry the corresponding conserved Noether charges,

$$J_1 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma \, \mathrm{Im}[\bar{Z}_1 \partial_\tau Z_1], \tag{25}$$

$$J_2 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma \, \mathrm{Im}[\bar{Z}_2 \partial_\tau Z_2], \tag{26}$$

which can be thought of as angular momenta in two orthogonal planes within S^3.

5.1. The ground state

On the gauge theory side the operator of lowest dimension for fixed charge J_1 is the chiral primary,

$$\hat{O} = \mathrm{Tr}_N[Z^{J_1}],$$

corresponding to the ferromagnetic ground state of the spin chain with exact scaling dimension $\Delta = J_1$. The corresponding solution in string theory is simply

$$Z_1 = X_1 + \mathrm{i}X_2 = \exp(\,\mathrm{i}t)$$

with $Z_2 = 0$ which gives, $\bar{Z}_1 \partial_t Z_1 = \mathrm{i} \Rightarrow J_1 = \sqrt{\lambda}\kappa = \Delta$. The solution $W = \exp(\,\mathrm{i}t)$ corresponds to a massless pointlike string moving at the speed of light around the equator of S^3 as shown in figure 22. This is also known as the BMN ground state of the string. In gauge theory we found that operators of definite scaling dimension were built out of excitations of the ferromagnetic ground state, i.e. magnons and their bound states. Here we will try to find the corresponding excitations on the string theory side of the correspondence.

On the gauge theory side the excitation picture was clearest in the limit of an infinite chain $L \to \infty$. The corresponding limit on the string theory side is [15]

$$\begin{aligned} &J_1 \to \infty, \qquad \Delta \to \infty \\ &\Delta - J_1 = \text{fixed}, \qquad \lambda = \text{fixed}, \qquad J_2 = \text{fixed}. \end{aligned} \tag{27}$$

As the energy density of the string is constant, the fact that $\Delta \to \infty$ implies that the string becomes *infinitely long* in the rescaled coordinate x introduced above. Keeping $\Delta - J$ finite restricts attention to states with a finite number of excitations or magnons. It requires that $Z_1(x, t)$ asymptotes to the ground state solution $Z_1 = \exp(\,\mathrm{i}t)$ as $x \to \infty$ while $Z_2 \to 0$.

5.2. Pohlmeyer reduction: S^2 case

To exhibit the integrability of the string equations of motion we will use a procedure known as Pohlmeyer reduction [22] to simplify the problem. We will begin by illustrating the procedure for the simpler case of string motion on a 2-sphere. Working on $S^2 \subset S^3$ we set $X_4 = 0$. In order to construct a string solution corresponding to an excitation of the BMN vacuum it will be useful to rewrite the string equation of motion and Virasoro constraint for the coordinates $\mathbf{X} = (X_1, X_2, X_3)$ in terms of a single nonlinear equation.

In terms of the $x_\pm$ coordinates, the string equation of motion coming from the variation of S_σ becomes

$$\partial_+\partial_-\mathbf{X} + (\partial_+\mathbf{X}\cdot\partial_-\mathbf{X})\,\mathbf{X} = 0. \tag{28}$$

The Virasoro constraint is

$$\partial_+\mathbf{X}\cdot\partial_+\mathbf{X} = \partial_-\mathbf{X}\cdot\partial_-\mathbf{X} = 1.$$

Thus $\partial_\pm\mathbf{X}$ are unit vectors in $\mathbb{R}^3$ and we can write

$$\partial_+\mathbf{X}\cdot\partial_-\mathbf{X} = \cos(\varphi)\,,$$

where $\varphi(x_+, x_-)$ is a real scalar field on the worldsheet. We can now describe dynamics of string in terms of $SO(3)$ invariant field φ. A short calculation shows that the string equation of motion (28) implies the following equation for φ,

$$\partial_+\partial_-\varphi + \sin(\varphi) = 0.$$

This is the *sine-Gordon equation* one of the most famous equations of mathematical physics. It is a nonlinear second-order PDE which has the property of *complete integrability* which we discuss below[7]. This integrability is manifested by certain hidden symmetries of the sine-Gordon (sG) equation corresponding to the so-called *Backlund transformation*. To illustrate this, suppose that $\varphi_0(x, t)$ satisfies the sine-Gordon equation,

$$\partial_+\partial_-\varphi_0 + \sin(\varphi_0) = 0.$$

We now define a new field configuration $\varphi_1(x, t)$ obeying

$$\frac{1}{2}\partial_+(\varphi_0 - \varphi_1) = \sin\left(\frac{\varphi_0 + \varphi_1}{2}\right) \tag{29}$$

$$-\frac{1}{2}\partial_-(\varphi_0 + \varphi_1) = \sin\left(\frac{\varphi_0 - \varphi_1}{2}\right). \tag{30}$$

[7] For integrability of the full classical string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ see [13, 14].

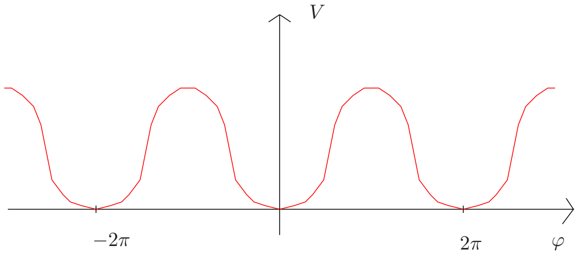


Figure 23. The sine-Gordon potential.

Acting on (29) with ∂_- gives

$$\begin{aligned}\frac{1}{2}\partial_-\partial_+(\varphi_0-\varphi_1) &= \partial_- \sin\left(\frac{\varphi_0+\varphi_1}{2}\right)\\ &= \frac{1}{2}\partial_-(\varphi_0+\varphi_1)\cos\left(\frac{\varphi_0+\varphi_1}{2}\right)\\ &= -\sin\left(\frac{\varphi_0-\varphi_1}{2}\right)\cos\left(\frac{\varphi_0+\varphi_1}{2}\right)\\ &= \frac{1}{2}\left[\sin(\varphi_1)-\sin(\varphi_0)\right].\end{aligned}$$

Thus

$$\partial_+\partial_-\varphi_1+\sin(\varphi_1) = \partial_+\partial_-\varphi_0+\sin(\varphi_0) = 0.$$

Thus, as $\varphi_0(x,t)$ satisfies the sG equation so does $\varphi_1(x,t)$. The Backlund transformation can be used to generate new solutions of sG. Finding φ_1 involves solving the first-order ODEs

$$\frac{1}{2}\partial_+(\varphi_0-\varphi_1) = \sin\left(\frac{\varphi_0+\varphi_1}{2}\right), \qquad -\frac{1}{2}\partial_-(\varphi_0+\varphi_1) = \sin\left(\frac{\varphi_0-\varphi_1}{2}\right),$$

which is an easier problem than solving the sG equation itself.

5.3. Solutions of the sine-Gordon equation

The sine-Gordon equation of motion arises naturally as the equation of motion for a scalar field $\varphi(x,t)$ in $(1+1)$ dimensions with potential energy,

$$V(\varphi) = 1-\cos(\varphi).$$

The vacua of the theory correspond to the minima of $V(\varphi)$ at $\varphi = 2\pi n$ for $n \in \mathbb{Z}$. We will study the theory with *vacuum boundary conditions*

$$\varphi(x,t) \to 2\pi n_\pm \quad \text{as} \quad x \to \pm\infty.$$

The space of solutions splits up into different sectors labelled by the integers $n_\pm$. More precisely we define a conserved *topological charge*,

$$Q = n_+ - n_-.$$

We will enumerate solutions in different sectors:

- $Q = 0$. The solution of minimum energy is simply the vacuum,

$$\varphi_V(x,t) \equiv 2\pi n$$

where $n_+ = n_- = n \in \mathbb{Z}$.

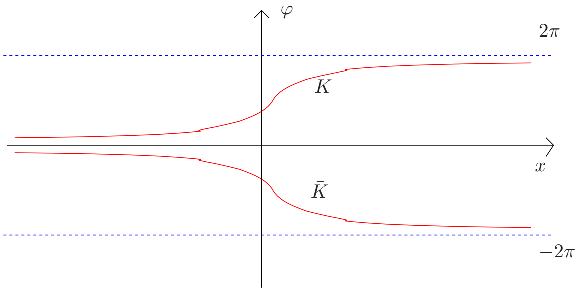


Figure 24. The sine-Gordon kink K and anti-kink $\bar{K}$.

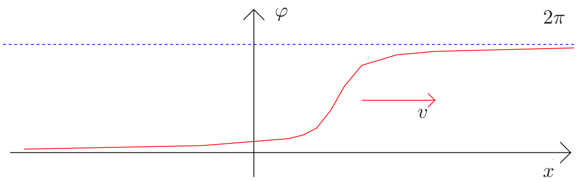


Figure 25. The sine-Gordon kink K moving at speed v.

- $Q = 1$. Applying the Backlund transformation to the vacuum solution φ_V, we find the static *kink* solution,
$$\varphi_K(x) = 4\tan^{-1}[\exp(x)]$$
which interpolates between adjacent vacua.
- Similarly or $Q = -1$ we have the corresponding *anti-kink* $\varphi_{\bar{K}}(x) = -\varphi_K(x)$. Both are shown in figure 24.
- Using the Lorentz invariance of the sG equation we can boost the static kink solution to obtain a solution corresponding to a kink moving with constant velocity v,
$$\varphi_K^{(v)}(x,t) = \varphi_K\left(\frac{x - vt}{\sqrt{1-v^2}}\right)$$
shown below.

5.4. Pohlmeyer reduction: S^3 case

We now return to the case of motion on S^3. As before, we must solve the equation
$$\partial_+\partial_-\mathbf{X} + (\partial_+\mathbf{X}\cdot\partial_-\mathbf{X})\,\mathbf{X} = 0 \tag{31}$$
together with the Virasoro constraint,
$$\partial_+\mathbf{X}\cdot\partial_+\mathbf{X} = \partial_-\mathbf{X}\cdot\partial_-\mathbf{X} = 1.$$

Following [22], we will begin by identifying the $SO(4)$ invariant combinations of the worldsheet fields $\mathbf{X}$ and their derivatives. As the first derivatives $\partial_\pm\mathbf{X}$ are unit vectors, we can again define a real scalar field $\phi(x,t)$ via the relation,
$$\cos\phi = \partial_+\mathbf{X}\cdot\partial_-\mathbf{X}. \tag{32}$$
Taking into account the constraint $|\mathbf{X}|^2 = 1$, we see that there are no other independent $SO(4)$ invariant quantities that can be constructed out of the fields and their first derivatives. At the

level of second derivatives we can construct two additional invariants which were absent in the case of the 2-sphere,

$$u\sin\phi = \partial_+^2\mathbf{X}\cdot\mathbf{K}, \qquad v\sin\phi = \partial_-^2\mathbf{X}\cdot\mathbf{K}, \tag{33}$$

where the components of vector $\mathbf{K}$ are given by $K_i = \epsilon_{ijkl}X_j\partial_+X_k\partial_-X_l$. The equations of motion for u, v and ϕ are derived in [22]. In fact the resulting equations imply that u and v are not independent and can be eliminated in favour of a new field $\chi(x,t)$ as

$$u = \partial_+\chi\tan\left(\frac{\phi}{2}\right), \qquad v = -\partial_-\chi\tan\left(\frac{\phi}{2}\right). \tag{34}$$

The equations of motion for χ and ϕ can then be written as

$$\partial_+\partial_-\phi + \sin\phi - \frac{\tan^2\left(\frac{\phi}{2}\right)}{\sin\phi}\partial_+\chi\partial_-\chi = 0, \tag{35}$$

$$\partial_+\partial_-\chi + \frac{1}{\sin\phi}(\partial_+\phi\partial_-\chi + \partial_-\phi\partial_+\chi) = 0. \tag{36}$$

In the special case of constant χ they reduce to the usual sine-Gordon equation for $\phi(x,t)$. Finally we can combine the real fields ϕ and χ to form a complex field $\psi = \sin(\phi/2)\exp(\,\mathrm{i}\chi/2)$, which obeys the equation,

$$\partial_+\partial_-\psi + \psi^*\frac{\partial_+\psi\partial_-\psi}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0. \tag{37}$$

Equation (37) is known as the *complex sine-Gordon equation* (CsG) [24]. Like the ordinary sG equation, it is completely integrable and has localized soliton solutions which undergo factorized scattering. The CsG equation is invariant under a global rotation of the phase of the complex field: $\psi \to \exp(\mathrm{i}\nu)\psi$, $\psi^* \to \exp(-\,\mathrm{i}\nu)\psi^*$. In addition to momentum and energy, CsG solitons carry the corresponding conserved $U(1)$ Noether charge. The most general one soliton solution to (37) is given by

$$\psi_{\text{1-soliton}} = \mathrm{e}^{\mathrm{i}\mu}\frac{\cos(\alpha)\exp(\,\mathrm{i}\sin(\alpha)T)}{\cosh(\cos(\alpha)(X-X_0))} \tag{38}$$

with

$$X = \cosh(\theta)x - \sinh(\theta)t, \qquad T = \cosh(\theta)t - \sinh(\theta)x. \tag{39}$$

The constant phase μ is irrelevant for our purposes as only the derivatives of the field χ affect the corresponding string solution. The parameter X_0 can be absorbed by a constant translation of the worldsheet coordinate x and we will set it to zero. The two remaining parameters of the solution are the rapidity θ of the soliton and an additional real number α which determines the $U(1)$ charge carried by the soliton.

Taking the limit $\alpha \to 0$, the field ϕ corresponding to the one-soliton solution (38) reduces to the kink solution of the ordinary sG equation.

5.5. The giant magnon

It remains to reconstruct the corresponding configuration of the string worldsheet fields $\mathbf{X}$ (or equivalently Z_1 and Z_2) corresponding to (38) for general values of the rapidity θ and rotation parameter α.

In this case we have

$$\partial_+\mathbf{X}\cdot\partial_-\mathbf{X} = \cos(\phi) = 1 - \frac{2\cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)}. \tag{40}$$

Hence the complex coordinates Z_1 and Z_2 must both solve the linear equation,

$$\frac{\partial^2 Z}{\partial t^2} - \frac{\partial^2 Z}{\partial x^2} + \left[1 - \frac{2\cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)}\right] Z = 0 \tag{41}$$

where as above $X = \cosh(\theta)x - \sinh(\theta)t$ and we impose the boundary conditions appropriate for a giant magnon with momentum p,

$$Z_1 \to \exp\left(\mathrm{i}t \pm \mathrm{i}\frac{p}{2}\right) Z_2 \to 0, \quad \text{as} \quad x \to \pm\infty. \tag{42}$$

As always the two complex fields obey the constraint $|Z_1|^2 + |Z_2|^2 = 1$. We will find unique solutions of the linear equation (41) obeying these conditions and then, for self-consistency, check that they correctly reproduce (40).

It is convenient to express the solution to (41) in terms of the boosted coordinates X and T. In terms of these variables $Z = Z[X, T]$ obeys

$$\frac{\partial^2 Z}{\partial T^2} - \frac{\partial^2 Z}{\partial X^2} + \left[1 - \frac{2\cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)}\right] Z = 0. \tag{43}$$

The problem now has the form of a Klein–Gordon equation describing the scattering of a relativistic particle in one spatial dimension incident on a static potential well. As usual the general solution of this equation can be written as a linear combination of 'stationary states' of the form,

$$Z_\omega = F_\omega(X)\exp(\mathrm{i}\omega T). \tag{44}$$

Rescaling the variables according to

$$\xi = \cos(\alpha)X, \qquad f(\xi) = F_\omega(X), \qquad \varepsilon = \frac{\sqrt{\omega^2 - 1}}{\cos(\alpha)}, \tag{45}$$

we find that the function $f(\xi)$ obeys the equation,

$$-\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{2}{\cosh^2(\xi)} f = \varepsilon^2 f. \tag{46}$$

Equation (46) coincides with the time-independent Schrödinger equation for a particle in (a special case of) the Rosen–Morse potential [25],

$$V(\xi) = \frac{-2}{\cosh^2(\xi)}. \tag{47}$$

The exact spectrum of this problem is known (see e.g. [26]). There is a single normalizable bound state with energy $\varepsilon^2 = -1$ and wavefunction,

$$f_{-1}(\xi) = \frac{1}{\cosh(\xi)} \tag{48}$$

and a continuum of scattering states with $\varepsilon^2 = k^2$ for $k > 0$ and wavefunctions,

$$f_{k^2}(\xi) = \exp(\mathrm{i}k\xi)(\tanh(\xi) - \mathrm{i}k) \tag{49}$$

with asymptotics

$$f_{k^2}(\xi) \to \exp\left(\mathrm{i}k\xi \pm \mathrm{i}\frac{\delta}{2}\right) \tag{50}$$

where the scattering phase shift is given as $\delta = 2\tan^{-1}(1/k)$.

The general solution to the original linear equation (41) can be constructed as a linear combination of these bound state and scattering wavefunctions. The particular solutions corresponding to the worldsheet fields Z_1 and Z_2 are singled out by the boundary conditions (42). In particular, the boundary condition (42) can only be matched by a solution corresponding to a single scattering mode $f_{k^2}(\xi)$,

$$Z_1 = c_1 f_{k^2}(\cos(\alpha)X)\exp(\,\mathrm{i}\omega_{k^2}T), \tag{51}$$

where $\omega_{k^2} = \sqrt{k^2\cos^2(\alpha)+1}$. We find that (42) is obeyed provided we set

$$k = \frac{\sinh(\theta)}{\cos(\alpha)} \tag{52}$$

which yields the magnon momentum $p = \delta = 2\tan^{-1}(1/k)$. The boundary condition (42) dictates that Z_2 decays at left and right infinity. This is only possible if we identify it with the solution corresponding to the unique normalizable bound state of the potential (46)

$$Z_2 = c_2 f_{-1}(\cos(\alpha)X)\exp(\,\mathrm{i}\omega_{-1}T) \tag{53}$$

with $\omega_{-1} = \sin(\alpha)$. Without loss of generality we can choose the constants c_1 and c_2 to be real. The condition $|Z_1|^2 + |Z_2|^2 = 1$ then yields

$$c_1 = c_2 = \frac{1}{\sqrt{1+k^2}}. \tag{54}$$

To summarize the above discussion the resulting string solution is

$$\begin{aligned} Z_1 &= \frac{1}{\sqrt{1+k^2}}(\tanh[\cos(\alpha)X] - \,\mathrm{i}k)\exp(\,\mathrm{i}t), \\ Z_2 &= \frac{1}{\sqrt{1+k^2}}\frac{1}{\cosh[\cos(\alpha)X]}\exp(\,\mathrm{i}\sin(\alpha)T), \end{aligned} \tag{55}$$

where X, T and k are defined in (39) and (52) above. One may easily check that this solution, in addition to obeying the string equation of motion (41) and boundary conditions (42), obeys the Virasoro constraints and satisfies the self-consistency condition (40).

The full solution (55) depends on two parameters: k and α. We can now evaluate the conserved charges $\Delta - J_1$ and J_2 as a function of these parameters,

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\frac{1}{1+k^2}\frac{\sqrt{1+k^2\cos^2(\alpha)}}{\cos(\alpha)}, \qquad J_2 = \frac{\sqrt{\lambda}}{\pi}\frac{1}{1+k^2}\tan(\alpha). \tag{56}$$

The magnon momentum is identified as $p = 2\tan^{-1}(1/k)$. Eliminating k and α we obtain the dispersion relation

$$\Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}. \tag{57}$$

As required by the boundary conditions the solution has finite $\Delta - J_1$ and the asymptotic behaviour is

$$Z_1 = X_1 + \,\mathrm{i}X_2 \to \exp\left(\,\mathrm{i}t \pm \mathrm{i}\frac{\Delta\phi}{2}\right)$$

as $x \to \pm\infty$ with $\Delta\phi = p = 2\tan^{-1}(1/k)$. The resulting solution thus corresponds to an open string with endpoints moving on the equator of S^2 at the speed of light. The angular separation between the endpoints is $\Delta\phi = p$. The solution, known as the Dyonic giant magnon, is shown below in figure 26. The fact that the solution describes an open string means that it does not actually appear in the spectrum of the IIB theory on $AdS_5 \times S^5$ which is a closed string theory. This is directly related to the fact that the $\mathcal{N} = 4$ theory does not contain a gauge-invariant operator corresponding to a single magnon state of the spin chain: it is forbidden by the cyclicity of the trace. As in the gauge theory case, we may construct

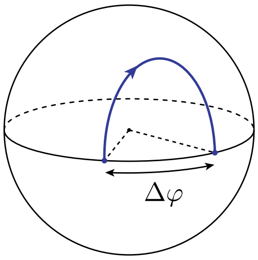


Figure 26. The giant magnon.

states with multiple giant magnons, which are allowed provided the total angular separation of the endpoints $\Delta\phi = \sum_{i=1}^{M} \Delta\phi_i$ vanishes modulus 2π, and the string is closed.

The time dependence of the solution (55) is also of interest. As in the original HM solution the constant phase rotation of Z_1 with exponent $\mathrm{i}t$ ensures that the endpoints of the string move on an equator of the 3-sphere at the speed of light. We can remove this dependence by changing coordinates from Z_1 to $\widetilde{Z}_1 = \exp(-\mathrm{i}t)Z_1$. In the new frame, the string configuration depends periodically on time through the t-dependence of Z_2. The period, $\mathcal{T}$, for this motion is the time for the solution to come back to itself up to a translation of the worldsheet coordinate x. From (55) we find

$$\mathcal{T} = 2\pi \frac{\cosh(\theta)}{\sin(\alpha)}. \tag{58}$$

As we have a periodic classical solution it is natural to define a corresponding action variable. A leading-order semiclassical quantization can then be performed by restricting the action variable to integral values according to the Bohr–Sommerfeld condition. Following [15], the action variable I is defined by the equation,

$$\mathrm{d}I = \frac{\mathcal{T}}{2\pi}\, \mathrm{d}(\Delta - J_1)|_p\,, \tag{59}$$

where the subscript p indicates that the differential is taken with fixed p. Using (56)–(58) we obtain simply $\mathrm{d}I = \mathrm{d}J_2$ which is consistent with the identification $I = J_2$. This is very natural as we expect the angular momentum J_2 to be integer valued in the quantum theory.

We can now compare the spectrum of excitations at weak and strong coupling. In the case $J_2 = 0$, the giant magnon dispersion relation becomes

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\left|\sin\left(\frac{p}{2}\right)\right|.$$

This is to be compared with the corresponding one-loop formula coming from the spin chain,

$$\Delta - J_1 = 1 + \frac{\lambda}{2\pi}\sin^2\left(\frac{p}{2}\right) + O(\lambda^2).$$

It is now known that the string theory and gauge theory results are the strong and weak coupling limits respectively of an exact dispersion relation [16] which gives

$$\Delta - J_1 = \sqrt{1 + \frac{\lambda}{\pi}\sin^2\left(\frac{p}{2}\right)}.$$

For the general case $J_2 \neq 0$, the semiclassical string theory result

$$\Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi}\sin^2\left(\frac{p}{2}\right)}$$

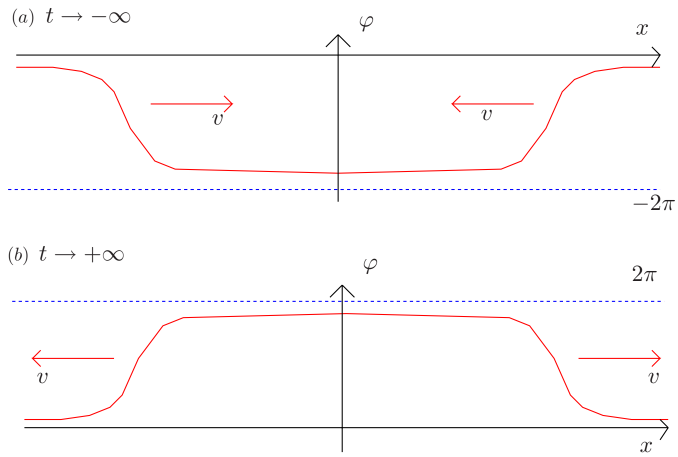


Figure 27. The sine-Gordon Kink $K - \bar{K}$ scattering solution.

can actually matches the exact bound state dispersion relation (21), provided we identify $J_2 = Q$ as the number of constituent magnons. The formula can also be expanded in powers of λ reproducing the one loop gauge theory result

$$\Delta - J_1 = Q + \frac{\lambda}{2\pi Q}\sin^2\left(\frac{p}{2}\right) + O(\lambda^2).$$

5.6. *Magnon scattering*

In this final section, we will examine how factorized scattering emerges on the string theory side of the correspondence. Integrability of sG and CsG equations manifested by the existence of exact analytic solutions describing soliton–soliton scattering. In the sG case studied above, these can be obtained by applying the Backlund transformation to the kink solutions. The resulting solution corresponds to the scattering of two giant magnons on the string.

$K\bar{K}$ scattering,

$$\varphi_{K\bar{K}}(x,t) = 4\tan^{-1}\left[\frac{\sinh\left(\frac{vt}{\sqrt{1-v^2}}\right)}{v\cosh\left(\frac{x}{\sqrt{1-v^2}}\right)}\right].$$

To understand the interpretation of this solution we consider the asymptotics at $t \to \pm\infty$ where it is well approximated as a superposition of a far-separated kink and anti-kink as shown in figure 27. More precisely we have

$$\varphi_{K\bar{K}}(x,t) \to \varphi_K\left(\frac{x + v\left(t \pm \frac{\Delta T}{2}\right)}{\sqrt{1-v^2}}\right) + \varphi_{\bar{K}}\left(\frac{x - v\left(t \pm \frac{\Delta T}{2}\right)}{\sqrt{1-v^2}}\right)$$

as $t \to \pm\infty$, with

$$\Delta T(v) = \frac{2}{v}\sqrt{1-v^2}\log(v).$$

Thus the solution describes an incident kink and anti-kink with velocities $\mp v$, which collide at $t = 0$. After the collision the velocities of the two solitons are unchanged and the only effect of the scattering is a time delay ΔT which depends on the velocity. A spacetime

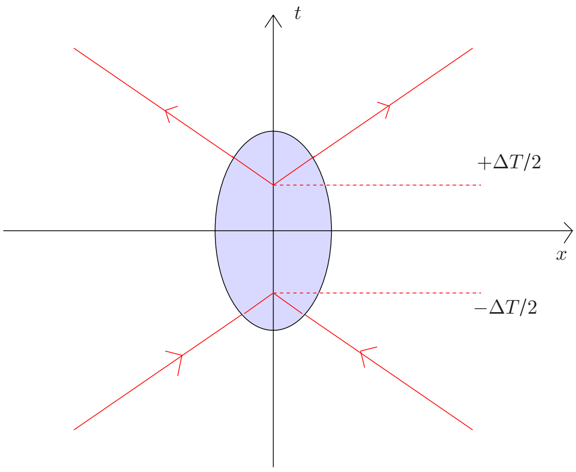


Figure 28. Spacetime picture of $K - \bar{K}$ scattering.

picture of the scattering is shown in figure 28. The integrability of the sG equation is reflected in the fact that solitons scatter with no loss of energy into radiation. In semi-classical quantization the leading order S-matrix for soliton scattering $\mathcal{S}(p_1, p_2) = \exp(\,\mathrm{i}\delta(p_1, p_2))$ can be calculated using the formula

$$\frac{\partial \delta}{\partial p} = \Delta T(v).$$

The resulting scattering phase

$$\delta_{sc}(p_1, p_2) = \frac{\sqrt{\lambda}}{\pi}\left[\cos\left(\frac{p_1}{2}\right) - \cos\left(\frac{p_2}{2}\right)\right]\log\left[\frac{\sin^2\left(\frac{p_1-p_2}{4}\right)}{\sin^2\left(\frac{p_1+p_2}{4}\right)}\right]$$

can be compared with the weak-coupling result described in the previous section,

$$\delta_{1-\text{loop}} = \frac{1}{\mathrm{i}}\log\left[\frac{\frac{1}{2}\cot\left(\frac{p_1}{2}\right) - \frac{1}{2}\cot\left(\frac{p_2}{2}\right) + \mathrm{i}}{\frac{1}{2}\cot\left(\frac{p_1}{2}\right) - \frac{1}{2}\cot\left(\frac{p_2}{2}\right) - \mathrm{i}}\right].$$

In fact, the exact S-matrix which interpolates between these limits is now known [17, 19, 20].

Similar solutions exist which describe multi-soliton scattering. These can be obtained by repeated applications of the Backlund transformation. Here integrability implies factorization into two-body scatterings as shown in figure 29. Total time delay experienced by kink 1,

$$\Delta T_{\text{total}} = \Delta T_{12} + \Delta T_{13}.$$

Semiclassical quantization then leads directly to a factorized S-matrix in accord with integrability. Because of this property, it suffices to compare the asymptotic spectrum and two-body S-matrix of gauge theory and string theory to demonstrate complete agreement between spectra (at least in the limit of large operators/strings).

Appendix A. The conformal group

The conformal group in four dimensions is $SO(4, 2)$ whose elements are real 6×6 matrices, M, of unit determinant obeying

$$M^T K M = K$$

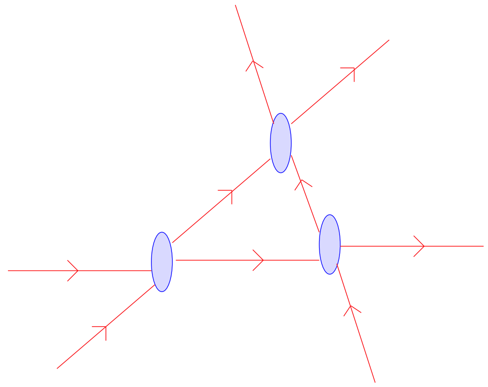
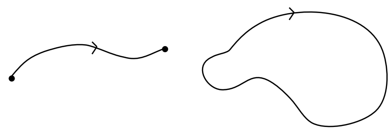


Figure 29. Three-soliton scattering.



(*a*) (*b*)

Figure 30. (*a*) An open string and (*b*) a closed string.

where K is the metric on $\mathbb{R}^{4,2}$,

$$K = \mathrm{diag}\,[-1, -1, +1, +1, +1, +1]\,.$$

The commutation relations for the generators given in the text are

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -\,\mathrm{i}\eta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations}\\
&[M_{\mu\nu}, P_\rho] = -\,\mathrm{i}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\\
&[M_{\mu\nu}, K_\rho] = -\,\mathrm{i}(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu)\\
&[P_\mu, K_\nu] = 2\mathrm{i}M_{\mu\nu} - 2\mathrm{i}\eta_{\mu\nu}D\\
&[M_{\mu\nu}, D] = 0, \qquad [D, P_\mu] = -\,\mathrm{i}P_\mu, \qquad [D, K_\mu] = \,\mathrm{i}K_\mu
\end{aligned} \tag{A.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

Appendix B. String theory basics

For a more detailed treatment see section 2 of [21] (in particular section 2.1).

String theory describes the dynamics of one-dimensional objects corresponding to either open or closed strings.

Here we will primarily be concerned with closed strings.

As the string moves it sweeps out a two-dimensional surface, Σ called a *worldsheet*. We can parametrize the worldsheet by introducing coordinates $\sigma \sim \sigma + 2\pi$ and τ as shown in figure 31.

It is often useful to work in terms of lightcone coordinates and the corresponding derivatives,

$$\sigma_\pm = \tau \pm \sigma, \qquad \partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma).$$

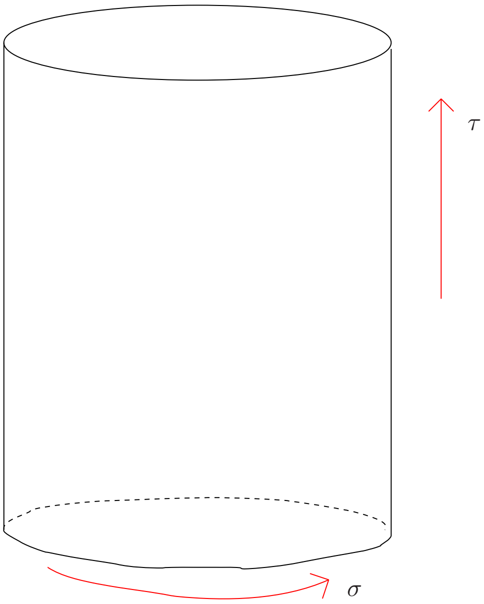


Figure A1. The string worldsheet.

The string moves in a $(D+1)$-dimensional spacetime $\mathcal{M}$ with coordinates X^M, $M = 0, 1, \ldots, D$ and metric tensor $G_{MN}(X)$.

The embedding of the string in spacetime is described by a map $\Sigma \to \mathcal{M}$, or, in coordinates,

$$(\sigma, \tau) \mapsto X^M(\sigma, \tau)$$

and string motion is described by the Polyakov–Brink–DiVechia–Howe action,

$$S = \frac{T}{2} \int d^2\sigma \, G_{MN}(X) \partial_+ X^M \partial_- X^N \tag{B.1}$$

subject to the closed string boundary conditions,

$$X^M(\sigma + 2\pi, \tau) = X^M(\sigma, \tau).$$

In these conventions the spacetime coordinates have the dimension of length $[X^M] = -1$ and the constant T, with $[T] = 2$, is the string tension. It is also conventional to write

$$T = \frac{1}{2\pi\alpha'},$$

where α' is the square of the characteristic length scale of the string.

There are two important additional conditions on string motion which come from the gauge invariance of the string associated with redefinitions of the worldsheet coordinates:

- Physical motions of the string must obey the *Virasoro constraint*,

$$G_{MN}(X) \partial_\pm X^M \partial_\pm X^N = 0.$$

- The action (B.1) itself has a residual gauge invariance under redefinitions of the coordinates of the form,

$$\sigma_+ \to f_+(\sigma_+) \qquad \sigma_- \to f_-(\sigma_-).$$

This gauge symmetry can be fixed in many different ways. Here we will focus on the *static gauge* where we impose the condition,

$$X^0 = \kappa \tau.$$

The classical equation of motion of the string is obtained by varying the action (B.1). In the special case of flat space, $\mathcal{M} = \mathbb{R}^{D,1}$, we replace $G_{MN}(X)$ by the $(D+1)$-dimensional Minkowski metric η_{MN} and the action becomes

$$S = \frac{T}{2} \int d^2\sigma \, \eta_{MN} \partial_+ X^M \partial_- X^N, \tag{B.2}$$

which is quadratic in the worldsheet fields $X^M(\sigma, \tau)$. Thus the resulting equation of motion is *linear*

$$\partial_+ \partial_- X^M = 0$$

for $M = 0, 1, \ldots, D$. This is trivially solved by writing each worldsheet field as a sum of left- and right-movers,

$$X^M(\sigma, \tau) = X^M_+(\sigma_+) + X^M_-(\sigma_-).$$

The general solution subject to closed string boundary conditions reads

$$X^M = x^M + 2\alpha' p^M + \sum_{n \neq 0} \frac{1}{n} \alpha^M_n \exp(-\,\mathrm{i} n \sigma_+) + \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^M_n \exp(-\mathrm{i} n \sigma_-)$$

- Here x^M and p^M correspond to the COM position and energy–momentum of the string in $\mathbb{R}^{D,1}$.
- The remaining parameters of the solution, α^M_n and $\tilde{\alpha}^M_n$ are known as left- and right-moving *oscillator coordinates*. The index n labels the different modes or harmonics of the string.

We now return to the general case of a curved spacetime manifold of radius R. The action

$$S = \frac{T}{2} \int d^2\sigma \, G_{MN}(X) \partial_+ X^M \partial_- X^N$$

is no longer quadratic in the fields and thus represents an interacting two-dimensional field theory. The corresponding equation of motion is therefore *nonlinear*. To identify the coupling of the interacting worldsheet theory we rescale the spacetime coordinates by a factor of the radius R,

$$X^M \rightarrow \tilde{X}^M = \frac{X^M}{R}$$

so that the rescaled coordinates are dimensionless: $\left[\tilde{X}^M\right] = 0$. In terms of the new coordinates the action reads

$$S = \frac{1}{g_\sigma^2} \int d^2\sigma \, G_{MN}(\tilde{X}) \partial_+ \tilde{X}^M \partial_- \tilde{X}^N$$

with coupling

$$g_\sigma = \sqrt{\frac{2}{TR}} = \sqrt{\frac{4\pi\alpha'}{R}}.$$

For the special case of $\mathcal{M} = \mathrm{AdS}_5 \times \mathrm{S}^5$, AdS/CFT provides the identification $g_\sigma^2 = 4\pi/\sqrt{\lambda}$.

References

[1] Maldacena J M 1998 The large N limit of superconformal field theories and supergravity *Adv. Theor. Math. Phys.* **2** 231

Maldacena J M 1999 The large N limit of superconformal field theories and supergravity *Int. J. Theor. Phys.* **38** 1113 (arXiv:hep-th/9711200)

[2] Aharony O, Gubser S S, Maldacena J M, Ooguri H and Oz Y 2000 Large N field theories, string theory and gravity *Phys. Rep.* **323** 183 (arXiv:hep-th/9905111)

[3] D'Hoker E and Freedman D Z 2002 Supersymmetric gauge theories and the AdS/CFT correspondence arXiv:hep-th/0201253

[4] Beisert N 2005 The dilatation operator of $N = 4$ super Yang–mills theory and integrability *Phys. Rep.* **405** 1 (arXiv:hep-th/0407277)

Plefka J 2005 Spinning strings and integrable spin chains in the AdS/CFT correspondence *Living Rev. Rel.* **8** 9 (arXiv:hep-th/0507136)

Tseytlin A A 2004 Semiclassical strings and AdS/CFT arXiv:hep-th/0409296

[5] Lipatov L N 1994 Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models *JETP Lett.* **59** 596

Lipatov L N 1994 Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models *Pisma Zh. Eksp. Teor. Fiz.* **59** 571

Faddeev L D and Korchemsky G P 1995 High-energy QCD as a completely integrable model *Phys. Lett.* B **342** 311 (arXiv:hep-th/9404173)

[6] Sohnius M and West P 1981 *Phys. Lett.* **100B** 45

[7] Seiberg N 1988 Supersymmetry and nonperturbative beta functions *Phys. Lett.* B **206** 75

[8] Minahan J A and Zarembo K 2003 The Bethe-ansatz for $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP03(2003)013 (arXiv:hep-th/0212208)

[9] Karbach M and Muller G 1998 Introduction to the Bethe ansatz: I arXiv:cond-mat/9809162

[10] Faddeev L D 1996 How algebraic Bethe ansatz works for integrable model arXiv:hep-th/9605187

[11] Babbit D and Thomas L 1978 *J. Math. Phys.* **19** 1699

[12] Metsaev R R and Tseytlin A A 1998 Type IIB superstring action in AdS(5) × S(5) background *Nucl. Phys.* B **533** 109 (arXiv:hep-th/9805028)

[13] Mandal G, Suryanarayana N V and Wadia S R 2002 Aspects of semiclassical strings in AdS(5) *Phys. Lett.* B **543** 81 (arXiv:hep-th/0206103)

[14] Bena I, Polchinski J and Roiban R 2004 Hidden symmetries of the AdS(5) × S**5 superstring *Phys. Rev.* D **69** 046002 (arXiv:hep-th/0305116)

[15] Hofman D M and Maldacena J M 2006 Giant magnons *J. Phys. A: Math. Gen.* **39** 13095 (arXiv:hep-th/0604135)

[16] Beisert N, Dippel V and Staudacher M 2004 A novel long range spin chain and planar $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP07(2004)075 (arXiv:hep-th/0405001)

[17] Staudacher M 2005 The factorized S-matrix of CFT/AdS *J. High Energy Phys.* JHEP05(2005)054 (arXiv:hep-th/0412188)

[18] Arutyunov G, Frolov S and Staudacher M 2004 Bethe ansatz for quantum strings *J. High Energy Phys.* JHEP10(2004)016 (arXiv:hep-th/0406256)

[19] Beisert N 2005 The su(2–2) dynamic S-matrix arXiv:hep-th/0511082

[20] Beisert N, Eden B and Staudacher M 2007 Transcendentality and crossing *J. Stat. Mech.* **0701** P021 (arXiv:hep-th/0610251)

[21] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory, vol 1: Introduction (Cambridge Monographs On Mathematical Physics)* (Cambridge: Cambridge University Press) p 469

[22] Pohlmeyer K 1976 Integrable Hamiltonian systems and interactions through quadratic constraints *Commun. Math. Phys.* **46** 207

[23] Dorey N 2006 Magnon bound states and the AdS/CFT correspondence *J. Phys. A: Math. Gen.* **39** 13119 (arXiv:hep-th/0604175)

[24] Lund F and Regge T 1976 Unified approach to strings and vortices with soliton solutions *Phys. Rev.* D **14** 1524

Lund F 1977 Example of a relativistic, completely integrable, Hamiltonian system *Phys. Rev. Lett.* **38** 1175

Getmanov B S 1979 Integrable model for the nonlinear complex scalar field with the nontrivial asymptotics of N-soliton solutions *Theor. Math. Phys.* **38** 124

Getmanov B S 1979 Integrable model for the nonlinear complex scalar field with the nontrivial asymptotics of N-soliton solutions *Teor. Mat. Fiz.* **38** 186

Getmanov B S 1982 Integrable two-dimensional Lorentz invariant nonlinear model of complex scalar field (complex sine-Gordon Ii) *Theor. Math. Phys.* **48** 572

Getmanov B S 1981 Integrable two-dimensional Lorentz invariant nonlinear model of complex scalar field (complex sine-Gordon Ii) *Teor. Mat. Fiz.* **48** 13

[25] Rosen N and Morse P 1932 *Phys. Rev.* **42** 210

[26] Landau L D and Lifshitz E M *Quantum Mechanics (Non-Relativistic Theory)* (Oxford: Pergamon) (p 73, problem 5 and p 80, problem 4)